# ECE 598 YM Homework 1

Assigned September 6, Due September 16

In this homework, you will gain familiarity with the basic algorithms and sufficient conditions for sparse recovery.

1. **Implementation of OMP.** Implement the basic orthogonal matching pursuit algorithm in Matlab [This was assigned on August 28]. Submit a printout of your code.

2. **$\ell^1$ as a linear program.** Show how the $\ell^1$-minimization problem

   $$\min \|x\|_1 \text{ subject to } y = Ax$$

   can be converted to a standard form linear program

   $$\min c^T z \quad \text{subject to} \quad Dz = e, \; Fz \leq g.$$

   Hint: first consider a scalar $\alpha$. Write down a linear program whose solution is $|\alpha|$. Generalize.

   Based on your conversion, write a Matlab function whose inputs are $y$ and $A$, and whose output is the minimum $\ell^1$-norm solution $\hat{x}_1$. You can either implement an LP solver, or use Matlab's `linprog`. Submit a printout of your code.

   For the remainder of this homework, please use your solutions to questions 1 and 2. For future homeworks, and for your course project, you can feel free to use

   `http://www.acm.caltech.edu/l1magic/`
   `http://sparselab.stanford.edu/`

   or any of the many of sparse recovery packages available at

   `http://www.dsp.ece.rice.edu/cs/#sof`.

3. **Mutual coherence for sparse recovery.** In lecture, we've discussed several sufficient conditions for sparse recovery involving the mutual coherence $\mu(A) \doteq \max_{i \neq j} \frac{\langle a_i, a_j \rangle}{\|a_i\|_2 \|a_j\|_2}$.

(a) In terms of $m$ and $n$, obtain [by searching the literature or deriving it yourself] a good lower bound on $\mu(A)$ that holds for all $m \times n$ matrices $A$.

(b) In class we gave a simple condition based on $\mu(A)$ guaranteeing that

- The sparsest solution to $y = Ax$ is unique.
- Both OMP and $\ell^1$-minimization recover this solution.

From part (a), as a function of $m$ and $n$, derive an upper bound on the number of nonzeros that the coherence-based results can guarantee we can recover.

In the limit as $m \to \infty$ and $m/n \to \delta < 1$, what is the order of growth of your bound? [Is it $O(m)$? $O(\log m)$?]

(c) Set $m = 100$, $n = 400$. Generate an $m \times n$ matrix with entries iid $\mathcal{N}(0, 1)$. Compute $\mu(A)$. Based on this value of $\mu(A)$, how many nonzeros can we guarantee to recover by either OMP or $\ell^1$?

Generate a sequence of $n$-dimensional vectors $x_1, x_2 \dots x_n$, by letting $x_k$ be the vector whose first $k$ entries are 1 and whose remaining $n - k$ entries are 0. For each $x_k$, set $y_k = Ax_k$. Use OMP and $\ell^1$-minimization to compute sparse solutions $\hat{x}_k$ to the system of equations $y_k = Ax_k$. Define the *breakdown point* as the smallest $k$ for which $\hat{x}_k \neq x_k$.[1] Compute the breakdown points of the two algorithms.

Repeat the above experiment several times. Plot and submit histograms of the following: 1. The number of nonzeros we can guarantee to recover based on $\mu(A)$. 2. The breakdown point of OMP. 3. The breakdown point of $\ell^1$-minimization.

(d) How does the actual performance of the algorithms compare to the bound from $\mu(A)$? How do the two algorithms' performances compare?

**READ BOTH, BUT ANSWER ONLY ONE OF THE FOLLOWING TWO PROBLEMS:**

4. **Another motivation for $\ell^1$.** In lecture, we mentioned that the $\ell^1$ norm is a natural convex surrogate for $\ell^0$ quasi-norm. In convex analysis, if $f(\cdot)$ is any function on a convex domain $\Gamma$, the *convex envelope* of $f$ is defined as the largest convex function $g$ such that $g(x) \leq f(x)$ for all $x \in \Gamma$.

   (a) Show that $\|\cdot\|_1$ is the convex envelope of $\|\cdot\|_0$ on the domain $\{x \,:\, \|x\|_\infty \leq 1\}$.

   (b) For a general function $f$, what is the relationship between the set of global minima of $f$ and the set of global minima of its convex envelope?

[1] Due to numerical imprecision, you will need to compare $\|\hat{x}_k - x_k\|$ to a small threshold.

(c) Think about (a) and (b). Why does this not imply that solving

$$\min \|x\|_1 \text{ subject to } y = Ax$$

always recovers the sparsest solution to the system $y = Ax$?

5. $\ell^0$ **solutions**. (a) Write a Matlab function solves the optimization problem

$$\min \|x\|_0 \quad \text{subject to} \quad y = Ax.$$

Hint: use Matlab's built-in function `nchoosek(1:n,k)` to enumerate all subsets of $k$ columns.

Submit your code.

(b) For an $m \times n$ Gaussian matrix $A$, what is $\mathrm{spark}(A)$?

(c) Set $n = 2m$, and choose $m$ small enough that your code from part (a) does not die when handed a problem of size $m \times n$. Generate an $m \times n$ Gaussian matrix $A$. Generate vectors $x_0$ of varying sparsity, from $\|x_0\|_0 = 1$ up to $\|x_0\|_0 = n$. Repeat several times. When at what sparsity $\|x_0\|_0$ does the solution given by your code start to differ from $x_0$?

(d) Think about your expression for $\mathrm{spark}(A)$ from part (b). Is the result of (c) surprising? Why or why not?