

# COMBINING GEOMETRY AND COMBINATORICS: A UNIFIED APPROACH TO SPARSE SIGNAL RECOVERY

R. BERINDE, A. C. GILBERT, P. INDYK, H. KARLOFF, AND M. J. STRAUSS

ABSTRACT. There are two main algorithmic approaches to sparse signal recovery: geometric and combinatorial. The geometric approach starts with a geometric constraint on the measurement matrix  $\Phi$  and then uses linear programming to decode information about  $x$  from  $\Phi x$ . The combinatorial approach constructs  $\Phi$  and a combinatorial decoding algorithm to match. We present a unified approach to these two classes of sparse signal recovery algorithms.

The unifying elements are the adjacency matrices of high-quality *unbalanced expanders*. We generalize the notion of *Restricted Isometry Property* (RIP), crucial to compressed sensing results for signal recovery, from the Euclidean norm to the  $\ell_p$  norm for  $p \approx 1$ , and then show that unbalanced expanders are essentially equivalent to RIP- $p$  matrices.

From known deterministic constructions for such matrices, we obtain new *deterministic* measurement matrix constructions and algorithms for signal recovery which, compared to previous deterministic algorithms, are superior in either the number of measurements or in noise tolerance.

## 1. INTRODUCTION

With the rise in high-speed data transmission and the exponential increase in data storage, it is imperative that we develop effective data compression techniques, techniques which accommodate both the volume and speed of data streams. A new approach to compressing  $n$ -dimensional vectors (or signals) begins with linear observations or measurements. For a signal  $x$ , its compressed representation is equal to  $\Phi x$ , where  $\Phi$  is a carefully chosen  $m \times n$  matrix,  $m \ll n$ , often chosen at random from some distribution. We call the vector  $\Phi x$  the *measurement vector* or a *sketch* of  $x$ . Although the dimension of  $\Phi x$  is much smaller than that of  $x$ , it retains many of the essential properties of  $x$ .

There are several reasons why linear compression or sketching is of interest. First, we can easily maintain a linear sketch  $\Phi x$  under linear updates to the signal  $x$ . For example, after incrementing the  $i$ -th coordinate  $x_i$ , we simply update the sketch as  $\Phi(x + e_i) = \Phi x + \Phi e_i$ . Similarly, we also easily obtain a sketch of a sum of two signals given the sketches for individual signals  $x$  and  $y$ , since  $\Phi(x + y) = \Phi x + \Phi y$ . Both properties are very useful in several computational areas, notably computing over data streams [AMS99, Mut03, Ind07], network measurement [EV03], query optimization and answering in databases [AMS99].

Another scenario where linear compression is of key importance is *compressed sensing* [CRT06, Don06a], a rapidly developing area in digital signal processing. In this setting,  $x$  is a physical

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Berinde is with the Department of Electrical Engineering and Computer Science, MIT. E-mail: [texel@mit.edu](mailto:texel@mit.edu). Gilbert is with the Department of Mathematics, The University of Michigan at Ann Arbor. E-mail: [annacg@umich.edu](mailto:annacg@umich.edu). Indyk is with the Computer Science and Artificial Intelligence Laboratory, MIT. E-mail: [indyk@theory.lcs.mit.edu](mailto:indyk@theory.lcs.mit.edu). Karloff is with AT&T Labs-Research. E-mail: [howard@research.att.com](mailto:howard@research.att.com). Strauss is with the Department of Mathematics and the Department of Electrical Engineering and Computer Science, The University of Michigan at Ann Arbor. E-mail: [martinjs@umich.edu](mailto:martinjs@umich.edu). ACG is an Alfred P. Sloan Research Fellow and has been supported in part by NSF DMS 0354600. MJS has been supported in part by NSF DMS 0354600 and NSF DMS 0510203. ACG and MJS have been partially supported by DARPA ONR N66001-06-1-2011. RB and PI are supported in part by David and Lucille Packard Fellowship. PI is partially supported by MADALGO (Center for Massive Data Algorithmics, funded by the Danish National Research Association).

signal one wishes to sense (e.g., an image obtained from a digital camera) and the linearity of the observations stems from a physical observation process. Rather than first observing a signal in its entirety and then compressing it, it may be less costly to sense the compressed version directly via a physical process. A camera “senses” the vector by computing a dot product with a number of pre-specified measurement vectors. See [TLW<sup>+</sup>06, DDT<sup>+</sup>08] for a prototype camera built using this framework. Other applications of linear sketching include database privacy [DMT07].

Although the sketch is considerably smaller than the original vector, we can still recover a large amount of information about  $x$ . See the surveys [Mut03, Ind07] on streaming and sublinear algorithms for a broad overview of the area. In this paper, we focus on retrieving a *sparse approximation*  $x_*$  of  $x$ . A vector is called *k-sparse* if it has at most  $k$  non-zero elements in the canonical basis (or, more generally,  $k$  non-zero coefficients in some basis  $B$ ). The goal of the sparse approximation is to find a vector  $x_*$  such that the  $\ell_p$  approximation error  $\|x - x_*\|_p$  is at most  $C > 0$  times the smallest possible  $\ell_q$  approximation error  $\|x - x'\|_q$ , where  $x'$  ranges over all  $k$ -sparse vectors. Note that the error  $\|x - x'\|_q$  is minimized when  $x'$  consists of the  $k$  largest (in magnitude) coefficients of  $x$ .

There are many algorithms for recovering sparse approximations (or their variants) of signals from their sketches. The early work on this topic includes the *algebraic* approach of [Man92] (cf. [GGI<sup>+</sup>02a]). Most of the known algorithms, however, can be roughly classified as either *combinatorial* or *geometric*.

**Combinatorial approach.** In the combinatorial approach, the measurement matrix  $\Phi$  is sparse and often binary. Typically, it is obtained from an adjacency matrix of a sparse bipartite random graph. The recovery algorithm proceeds by iteratively, identifying and eliminating “large” coefficients<sup>1</sup> of the vector  $x$ . The identification uses non-adaptive binary search techniques. Examples of combinatorial sketching and recovery algorithms include [GGI<sup>+</sup>02b, CCFC02, CM04, GKMS03, DWB05, SBB06b, SBB06a, CM06, GSTV06, GSTV07, Ind08, XH07] and others.

The typical advantages of the combinatorial approach include fast recovery, often sub-linear in the signal length  $n$  if  $k \ll n$  and fast and incremental (under coordinate updates) computation of the sketch vector  $\Phi x$ . In addition, it is possible to construct efficient (albeit suboptimal) measurement matrices *explicitly*, at least for simple type of signals. For example, it is known [Ind08, XH07] how to explicitly construct matrices with  $k2^{(\log \log n)^{O(1)}}$  measurements, for signals  $x$  that are exactly  $k$ -sparse. The main disadvantage of the approach is the suboptimal sketch length.

**Geometric approach.** This approach was first proposed in the papers [CRT06, Don06a] and has been extensively investigated since then (see [Gro06] for a bibliography). In this setting, the matrix  $\Phi$  is dense, with at least a constant fraction of non-zero entries. Typically, each row of the matrix is independently selected from a sub-exponential  $n$ -dimensional distribution, such as Gaussian or Bernoulli. The key property of the matrix  $\Phi$  which yields efficient recovery algorithms is the *Restricted Isometry Property* [CRT06], which requires that for any  $k$ -sparse vector  $x$  we have  $\|\Phi x\|_2 = (1 \pm \delta)\|x\|_2$ . If a matrix  $\Phi$  satisfies this property, then the recovery process can be accomplished by finding a vector  $x_*$  using the following linear program:

$$\min \|x_*\|_1 \text{ subject to } \Phi x_* = \Phi x. \tag{P1}$$

The advantages of the geometric approach include a small number of measurements ( $O(k \log(n/2k))$  for Gaussian matrices and  $O(k \log^{O(1)} n)$  for Fourier matrices) and resiliency to measurement errors<sup>2</sup>. The main disadvantage is the running time of the recovery procedure, which involves solving a linear program with  $n$  variables and  $n + m$  constraints. The computation of the sketch  $\Phi x$  can be

<sup>1</sup>In the non-sketching world, such methods algorithms are often called “weak greedy algorithms”, and have been studied thoroughly by Temlyakov [Tem02]

<sup>2</sup>Historically, the geometric approach resulted also in the first *deterministic* or *uniform* recovery algorithms, where a fixed matrix  $\Phi$  was guaranteed to work for *all* signals  $x$ . In contrast, the early combinatorial sketching algorithms only guaranteed  $1 - 1/n$  probability of correctness for *each* signal  $x$ . However, the papers [GSTV06, GSTV07] showed that combinatorial algorithms can achieve deterministic or uniform guarantees as well.

done efficiently for some matrices (e.g., Fourier); however, an efficient sketch update is not possible. In addition, the problem of finding an explicit construction of efficient matrices satisfying the RIP property is open [Tao07]; the best known explicit construction [DeV07] yields  $\Omega(k^2)$  measurements.

**Connections.** There has been some recent progress in obtaining the advantages of both approaches by decoupling the algorithmic and combinatorial aspects of the problem. Specifically, the papers [NV07, DM08, NT08] show that one can use *greedy* methods for data compressed using *dense* matrices satisfying the RIP property. Similarly [GLR08], using the results of [KT07], show that sketches from (somewhat) sparse matrices can be recovered using linear programming.

The best results (up to  $O(\cdot)$  constants) obtained prior to this work are shown in Figure 1<sup>3</sup>. We ignore some aspects of the algorithms, such as explicitness or universality of the measurement matrices. Furthermore, we present only the algorithms that work for arbitrary vectors  $x$ , while many other results are known for the case where the vector  $x$  itself is exactly  $k$ -sparse; e.g., see [TG05, DWB05, SBB06b, Don06a, XH07]. The columns describe: citation; whether the recovery algorithms hold with high probability for All signals or for Each signal; sketch length; time to compute  $\Phi x$  given  $x$ ; time to update  $\Phi x$  after incrementing one of the coordinates of  $x$ ; time<sup>4</sup> to recover an approximation of  $x$  given  $\Phi x$ ; approximation guarantee; and whether the algorithm is robust to noisy measurements. In the approximation error column,  $\ell_p \leq A\ell_q$  means that the algorithm returns a vector  $x_*$  such that  $\|x - x_*\|_p \leq A \min_{x'} \|x - x'\|_q$ , where  $x'$  ranges over all  $k$ -sparse vectors. In [CDD06], the authors show that an approximation guarantee of the form “ $\ell_2 \leq \frac{C}{k^{1/2}}\ell_1$ ” implies a “ $\ell_1 \leq (1 + O(C))\ell_1$ ” guarantee, and that it is impossible to achieve “ $\ell_2 \leq C\ell_2$ ” deterministically (or for all signals simultaneously) unless the number of measurements is  $\Omega(n)$ . The parameters  $C > 1$ ,  $c \geq 2$  and  $a > 0$  denote absolute constants, possibly different in each row. We assume that  $k < n/2$ .

In addition, in Figure 2 we present very recent results discovered during the course of our research. Some of the running times of the algorithms depend on the “precision parameter”  $D$ , which is always bounded from the above by  $\|x\|_2$  if the coordinates of  $x$  are integers.

**1.1. Our results.** In this paper we give a sequence of results which indicate that the combinatorial and geometric approaches are, in a rigorous sense, different manifestations of a common underlying phenomenon. This enables us to achieve a unifying perspective on both approaches, as well as obtaining several new concrete algorithmic results.

We consider matrices which are *binary* and *sparse*; i.e., they have only a small number  $d$  of ones in each column, and all the other entries are equal to zero. It has been shown recently [Cha08] that such matrices cannot satisfy the RIP property with parameters  $k$  and  $\delta$ , unless the number of rows is  $\Omega(k^2)$ . Our first result is that, nevertheless, such matrices satisfy a different form of the RIP property, that we call the *RIP- $p$  property*, where the  $\ell_2$  norm is replaced by the  $\ell_p$  norm. Formally, the matrix  $\Phi$  satisfies RIP <sub>$p,k,\delta$</sub>  property if for any  $k$ -sparse vector  $x$  we have  $\|\Phi x\|_p = (1 \pm \delta)\|x\|_p$ . In particular, we show that this property holds for  $1 \leq p \leq 1 + O(1)/\log n$  if the matrix  $\Phi$  is an adjacency matrix of a high-quality *unbalanced expander graph*. Thus we have a large class of natural such measurement matrices. We also exhibit an RIP-2 matrix that is not an RIP-1 matrix, so that, with [Cha08], we conclude that these two conditions are incomparable—neither one is stronger than the other.

<sup>3</sup>Some of the papers, notably [CM04], are focused on a somewhat different formulation of the problem. However, it is known that the guarantees presented in the table hold for those algorithms as well. See Lecture 4 in [Ind07] for a more detailed discussion.

<sup>4</sup>In the decoding time column LP=LP( $n, m, T$ ) denotes the time needed to solve a linear program defined by an  $m \times n$  matrix  $\Phi$  which supports matrix-vector multiplication in time  $T$ . Heuristic arguments indicate that LP( $n, m, T$ )  $\approx \sqrt{n}T$  if the interior-point method is employed. In addition, the paper [NV07] does not discuss the running time of the algorithm. Our bound is obtained by multiplying the number of algorithm iterations (i.e.,  $k$ ) by the number of entries in the matrix  $\Phi$  (i.e.,  $nk \log^c n$ ). See [NT08] for an in-depth discussion of the running times of OMP-based procedures.

Paper	A/E	Sketch length	Encode time	Column sparsity/ Update time	Decode time	Approx. error	Noise
[CCFC02, CM06]	E	$k \log^c n$	$n \log^c n$	$\log^c n$	$k \log^c n$	$\ell_2 \leq C \ell_1$	
	E	$k \log n$	$n \log n$	$\log n$	$n \log n$	$\ell_2 \leq C \ell_1$	
[CM04]	E	$k \log^c n$	$n \log^c n$	$\log^c n$	$k \log^c n$	$\ell_1 \leq C \ell_1$	
	E	$k \log n$	$n \log n$	$\log n$	$n \log n$	$\ell_1 \leq C \ell_1$	
[CRT06]	A	$k \log(n/k)$	$nk \log(n/k)$	$k \log(n/k)$	LP	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	Y
	A	$k \log^c n$	$n \log n$	$k \log^c n$	LP	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	Y
[GSTV06]	A	$k \log^c n$	$n \log^c n$	$\log^c n$	$k \log^c n$	$\ell_1 \leq C \log n \ell_1$	Y
[GSTV07]	A	$k \log^c n$	$n \log^c n$	$\log^c n$	$k^2 \log^c n$	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	
[NV07]	A	$k \log(n/k)$	$nk \log(n/k)$	$k \log(n/k)$	$nk^2 \log^c n$	$\ell_2 \leq \frac{C(\log n)^{1/2}}{k^{1/2}} \ell_1$	Y
	A	$k \log^c n$	$n \log n$	$k \log^c n$	$nk^2 \log^c n$	$\ell_2 \leq \frac{C(\log n)^{1/2}}{k^{1/2}} \ell_1$	Y
[GLR08] (k “large”)	A	$k(\log n)^c \log \log n$	$kn^{1-a}$	$n^{1-a}$	LP	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	
This paper	A	$k \log(n/k)$	$n \log(n/k)$	$\log(n/k)$	LP	$\ell_1 \leq C \ell_1$	Y

FIGURE 1. Summary of the best prior results.

Paper	A/E	Sketch length	Encode time	Update time	Decode time	Approx. error	Noise
[DM08]	A	$k \log(n/k)$	$nk \log(n/k)$	$k \log(n/k)$	$nk \log(n/k) \log D$	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	Y
[NT08]	A	$k \log(n/k)$	$nk \log(n/k)$	$k \log(n/k)$	$nk \log(n/k) \log D$	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	Y
	A	$k \log^c n$	$n \log n$	$k \log^c n$	$n \log n \log D$	$\ell_2 \leq \frac{C}{k^{1/2}} \ell_1$	Y
[IR08]	A	$k \log(n/k)$	$n \log(n/k)$	$\log(n/k)$	$n \log(n/k)$	$\ell_1 \leq C \ell_1$	Y

FIGURE 2. Recent work.

**Theorem 1.** Consider any  $m \times n$  matrix  $\Phi$  that is the adjacency matrix of an  $(k, \epsilon)$ -unbalanced expander  $G = (A, B, E)$ ,  $|A| = n$ ,  $|B| = m$ , with left degree  $d$ , such that  $1/\epsilon, d$  are smaller than  $n$ . Then the scaled matrix  $\Phi/d^{1/p}$  satisfies the  $\text{RIP}_{p,k,\delta}$  property, for  $1 \leq p \leq 1 + 1/\log n$  and  $\delta = C\epsilon$  for some absolute constant  $C > 1$ .

The fact that the unbalanced expanders yield matrices with RIP- $p$  property is not an accident. In particular, we show in Section 2 that any binary matrix  $\Phi$  in which each column has  $d$  ones<sup>5</sup> and which satisfies RIP-1 property with proper parameters, must be an adjacency matrix of a good unbalanced expander. That is, an RIP- $p$  matrix and the adjacency matrix of an unbalanced expander are essentially equivalent. Therefore, RIP-1 provides an interesting “analytic” formulation of expansion for unbalanced graphs. Also, without significantly improved explicit constructions of unbalanced expanders with parameters that match the probabilistic bounds (a longstanding open problem), we do not expect significant improvements in the explicit constructions of RIP-1 matrices.

<sup>5</sup>In fact, the latter assumption can be removed without loss of generality. The reason is that, from the RIP-1 property alone, it follows that each column must have roughly the same number of ones. The slight unbalance in the number of ones does not affect our results much; however, it does complicate the notation somewhat. As a result, we decided to keep this assumption throughout the paper.

**Theorem 2.** Consider any  $m \times n$  binary matrix  $\Phi$  such that each column has exactly  $d$  ones. If for some scaling factor  $S > 0$  the matrix  $S\Phi$  satisfies the  $\text{RIP}_{1,s,\delta}$  property, then the matrix  $\Phi$  is an adjacency matrix of an  $(s, \epsilon)$ -unbalanced expander, for

$$\epsilon = \left(1 - \frac{1}{1 + \delta}\right) / (2 - \sqrt{2}).$$

In the next step in Section 3, we show that the RIP-1 property of a binary matrix (or, equivalently, the expansion property) alone suffices to guarantee that the linear program  $P_1$  recovers a good sparse approximation. In particular, we show the following

**Theorem 3.** Let  $\Phi$  be an  $m \times n$  matrix of an unbalanced  $(2k, \epsilon)$ -expander. Let  $\alpha(\epsilon) = (2\epsilon)/(1 - 2\epsilon)$ . Consider any two vectors  $x, x_*$ , such that  $\Phi x = \Phi x_*$ , and  $\|x_*\|_1 \leq \|x\|_1$ . Then

$$\|x - x_*\|_1 \leq 2/(1 - 2\alpha(\epsilon)) \cdot \|x - x_k\|_1.$$

where  $x_k$  is the optimal  $k$ -term representation for  $x$ .

We also provide a noise-resilient version of the theorem; see Section 3 for details.

By combining Theorem 3 with the best known probabilistic constructions of expanders (Section 2) we obtain a scheme for sparse approximation recovery with parameters as in Figure 1. The scheme achieves the best known bounds for several parameters: the scheme is deterministic (one matrix works for all vectors  $x$ ), the number of measurements is  $O(k \log(n/k))$ , the update time is  $O(\log(n/k))$  and the encoding time is  $O(n \log(n/k))$ . In particular, this provides the first known scheme which achieves the best known measurement and encoding time bounds *simultaneously*. In contrast, the Gaussian and Fourier matrices are known to achieve only one optimal bound at a time. The fast encoding time also speeds-up the LP decoding, given that the linear program is typically solved using the interior-point method, which repeatedly performs matrix-vector multiplications. In addition to theoretical guarantees, random sparse matrices offer an attractive empirical performance. We show in Section 4 that the empirical behavior of binary sparse matrices with LP decoding is consistent with the analytic performance of Gaussian random matrices.

In the final part of the paper, we show that adjacency matrices of unbalanced expanders can be augmented to facilitate sub-linear time combinatorial recovery. This fact has been implicit in the earlier work [GSTV07, Ind08]; here we verify that indeed the expansion property is the sufficient condition guaranteeing correctness of those algorithms. As a result, we obtain an explicit construction of matrices with  $O(k2^{(\log \log n)^{O(1)}})$  rows that are amenable to a sublinear decoding algorithm for all vectors (similar to that in [GSTV07]). Previous explicit constructions for sublinear time algorithms either had  $\Omega(k^2)$  rows [CM06] or had  $O(k2^{(\log \log n)^{O(1)}})$  rows [Ind08, XH07] but were restricted to  $k$ -sparse signals or their slight generalizations. An additional (and somewhat unexpected) consequence is that the algorithm of [Ind08] is simple, effectively mimicking the well-known “parallel bit-flip” algorithm for decoding low-density parity-check codes.

**Theorem 4.** Let  $\epsilon > 0$  be a fixed constant, and  $p = 1 + 1/\log n$ . Consider  $x \in \mathbb{R}^n$  and a sparsity parameter  $k$ . There is a measurement matrix  $\Psi$ , which we can construct explicitly or randomly, and an algorithm  $\text{HHS}(p)$  that, given measurements  $v = \Psi x$  of  $x$ , returns an approximation  $\hat{x}$  of  $x$  with  $O(k/\epsilon)$  nonzero entries. The approximation satisfies

$$\|x - \hat{x}\|_p \leq \epsilon k^{1/p-1} \|x - x_k\|_1.$$

where  $x_k$  is the optimal  $k$ -term representation for  $x$ .

Figure 3 summarizes the connections among all of our results. We show the relationship between the combinatorial and geometric approaches to sparse signal recovery

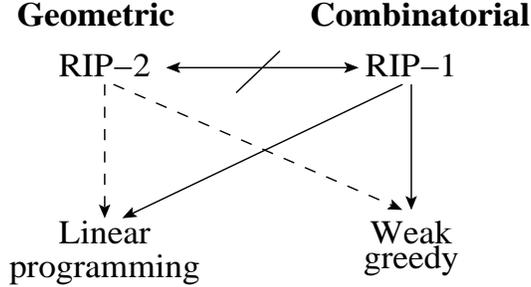


FIGURE 3. The above diagram captures the relations between the combinatorial and geometric approaches and the two main classes of decoding algorithms. Connections established in prior work are shown with dashed lines. Our work connects both approaches, with the ultimate goal of obtaining the “best of both worlds.”

## 2. UNBALANCED EXPANDERS AND RIP MATRICES

**2.1. Unbalanced expanders.** In this section we show that RIP- $p$  matrices for  $p \approx 1$  can be constructed using high-quality expanders. The formal definition of the latter is as follows.

**Definition 5.** A  $(k, \epsilon)$ -unbalanced expander is a bipartite simple graph  $G = (A, B, E)$  with left degree  $d$  such that for any  $X \subset A$  with  $|X| \leq k$ , the set of neighbors  $N(X)$  of  $X$  has size  $|N(X)| \geq (1 - \epsilon)d|X|$ .

In constructing such graphs, our goal is to make  $|B|$ ,  $d$ , and  $\epsilon$  as small as possible, while making  $k$  as close to  $|B|$  as possible.

The following well-known proposition can be shown using the probabilistic method.

**Proposition 6.** For any  $n/2 \geq k \geq 1$ ,  $\epsilon > 0$ , there exists a  $(k, \epsilon)$ -unbalanced expander with left degree  $d = O(\log(n/k)/\epsilon)$  and right set size  $O(kd/\epsilon) = O(k \log(n/k)/\epsilon^2)$ .

**Proposition 7.** For any  $n \geq k \geq 1$  and  $\epsilon > 0$ , one can explicitly construct a  $(k, \epsilon)$ -unbalanced expander with left degree  $d = 2^{O(\log(\log(n/\epsilon)))^3}$ , left set size  $n$  and right set size  $m = kd/\epsilon^{O(1)}$ .

*Proof.* The construction is given in [CRVW02], Theorem 7.3. Note that the theorem refers to notion of *lossless conductors*, which is equivalent to unbalanced expanders, modulo representing all relevant parameters (set sizes, degree, etc.) in the log-scale. After an additional  $O(nd)$ -time postprocessing, we can ensure that the graph is simple; i.e., it contains no duplicate edges.  $\square$

## 2.2. Construction of RIP matrices.

**Definition 8.** An  $m \times n$  matrix  $\Phi$  is said to satisfy  $\text{RIP}_{p,k,\delta}$  if, for any  $k$ -sparse vector  $x$ , we have

$$\|x\|_p \leq \|\Phi x\|_p \leq (1 + \delta) \|x\|_p.$$

Observe that the definitions of  $\text{RIP}_{1,k,\delta}$  and  $\text{RIP}_{2,k,\delta}$  matrices are incomparable. In what follows below, we present sparse binary matrices with  $O(k \log(n/k))$  rows that are  $\text{RIP}_{1,k,\delta}$ ; it has been shown recently [Cha08] that sparse binary matrices cannot be  $\text{RIP}_{2,k,\delta}$  unless the number of rows is  $\Omega(k^2)$ . In the other direction, consider an appropriately scaled random Gaussian matrix  $G$  of  $R \approx k \log(n)$  rows. Such a matrix is known to be  $\text{RIP}_{2,k,\delta}$ . To see that this matrix is *not*  $\text{RIP}_{1,k,\delta}$ , consider the signal  $x$  consisting of all zeros except a single 1 and the signal  $y$  consisting of all zeros except  $k$  terms with coefficient  $1/k$ . Then  $\|x\|_1 = \|y\|_1$  but  $\|Gx\|_1 \approx \sqrt{k} \|Gy\|_1$ .

**Theorem 1** Consider any  $m \times n$  matrix  $\Phi$  that is the adjacency matrix of an  $(k, \epsilon)$ -unbalanced expander  $G = (A, B, E)$  with left degree  $d$ , such that  $1/\epsilon, d$  are smaller than  $n$ . Then the scaled matrix  $\Phi/d^{1/p}$  satisfies the  $\text{RIP}_{p,k,\delta}$  property, for  $1 \leq p \leq 1 + 1/\log n$  and  $\delta = C\epsilon$  for some absolute constant  $C > 1$ .

*Proof.* Let  $x \in \mathbb{R}^n$  be a  $k$ -sparse vector. Without loss of generality, we assume that the coordinates of  $x$  are ordered such that  $|x_1| \geq \dots \geq |x_n|$ .

The proof proceeds in two stages. In the first part, we show that the theorem holds for the case of  $p = 1$ . In the second part, we extend the theorem to the case where  $p$  is slightly larger than 1.

**The case of  $p = 1$ .** We order the edges  $e_t = (i_t, j_t)$ ,  $t = 1 \dots dn$  of  $G$  in a lexicographic manner. It is helpful to imagine that the edges  $e_1, e_2 \dots$  of  $E$  are being added to the (initially empty) graph. An edge  $e_t = (i_t, j_t)$  causes a *collision* if there exists an earlier edge  $e_s = (i_s, j_s)$ ,  $s < t$ , such that  $j_t = j_s$ . We define  $E'$  to be the set of edges which do *not* cause collisions, and  $E'' = E - E'$ .

**Lemma 9.** *We have*

$$\sum_{(i,j) \in E''} |x_i| \leq \epsilon d \|x\|_1$$

*Proof.* For each  $t = 1 \dots dn$ , we use an indicator variable  $r_t \in \{0, 1\}$ , such that  $r_t = 1$  iff  $e_t \in E''$ . Define a vector  $z \in \mathbb{R}^{dn}$  such that  $z_t = |x_{i_t}|$ . Observe that

$$\sum_{(i,j) \in E''} |x_i| = \sum_{e_t = (i_t, j_t) \in E} r_t |x_{i_t}| = r \cdot z$$

To upper bound the latter quantity, observe that the vectors satisfy the following constraints:

- The vector  $z$  is non-negative.
- The coordinates of  $z$  are monotonically non-increasing.
- For each *prefix set*  $P_i = \{1 \dots di\}$ ,  $i \leq k$ , we have  $\|r|_{P_i}\|_1 \leq \epsilon di$  - this follows from the expansion properties of the graph  $G$ .
- $r|_{P_1} = 0$ , since the graph is simple.

It is now immediate that for any  $r, z$  satisfying the above constraints, we have  $r \cdot z \leq \|z\|_1 \epsilon$ . Since  $\|z\|_1 = d \|x\|_1$ , the lemma follows.  $\square$

Lemma 9 immediately implies that  $\|\Phi x\|_1 \geq d \|x\|_1 (1 - 2\epsilon)$ . Since for any  $x$  we have  $\|\Phi x\|_1 \leq d \|x\|_1$ , it follows that  $\Phi/d$  satisfies the  $\text{RIP}_{1,k,2\epsilon}$  property.

**The case of  $p \leq 1 + 1/\log n$ .** Let  $u = \Phi x$ . We will show that if  $\epsilon$  is small enough, then the value of  $\|u\|_p^p$  is close to  $d \|x\|_p^p$ .

We start from a few useful technical claims.

**Claim 10.** *For any  $a, b \in \mathbb{R}$ , we have  $|(a + b)|^p \geq |a|^p - p|a|^{p-1}|b|$ .*

**Claim 11.** *For any  $a \in \mathbb{R}$  and  $|b| \leq |a|$ , we have  $|(a + b)|^p \leq |a|^p + p|a|^{p-1}|b|$ .*

For any  $j \in B$ , define  $u'_j = x_i$  if  $(i, j) \in E'$ , and  $u'_j = 0$  otherwise. Also, define  $u''_j = u_j - u'_j$ .

**Claim 12.** *We have  $\sum_j |u''_j|^p = \Theta(\epsilon) d \|x\|_p^p$ .*

*Proof.* By Lemma 9 we know that  $\sum_j |u'_j| = \sum_{(i,j) \in E'} |x_i| \geq (1 - \epsilon) d \|x\|_1$ . Therefore

$$\sum_j |u''_j|^p \leq \left( \sum_j |u'_j| \right)^p \leq (\epsilon d \|x\|_1)^p \leq \epsilon d^p n^{(1-1/p)p} \|x\|_p^p = \Theta(\epsilon) d \|x\|_p^p$$

$\square$

**Lemma 13.** *We have  $\|u\|_p^p \geq (1 - \Theta(\epsilon)) d \|x\|_p^p$ .*

*Proof.* By Claim 10 we know that

$$\|u\|_p^p = \sum_j |u'_j + u''_j|^p \geq \sum_j |u'_j|^p - \sum_j p |u'_j|^{p-1} |u''_j|$$

Define the set  $S = \{j : \epsilon|u_j''| \leq |u_j''|\}$ . We have

$$\begin{aligned} \sum_j p|u_j|^{p-1}|u_j''| &= \sum_{j \in S} p|u_j|^{p-1}|u_j''| + \sum_{j \notin S} p|u_j|^{p-1}|u_j''| \leq (1/\epsilon)^{p-1} p \sum_{j \in S} |u_j''|^p + p \sum_{j \notin S} \epsilon|u_j''|^p \\ &\leq \Theta \left[ \sum_j |u_j''|^p + \epsilon \sum_j |u_j''|^p \right] \leq \Theta(\epsilon) d \|x\|_p^p + \epsilon \|u''\|_p^p \end{aligned}$$

Therefore

$$\|u\|_p^p \geq \|u'\|_p^p - \Theta(1)d\epsilon \|x\|_p^p - \Theta(\epsilon) \|u''\|_p^p = \|u'\|_p^p (1 - \Theta(\epsilon)) - \Theta(1)d\epsilon \|x\|_p^p.$$

It suffices to bound  $\|u'\|_p^p$  from below.

$$\begin{aligned} \|u'\|_p^p &\geq d \|x\|_p^p - \sum_{(i,j) \in E''} |x_i|^p \geq d \|x\|_p^p - \left( \sum_{(i,j) \in E''} |x_i| \right)^p \\ &\geq d \|x\|_p^p - \Theta(\epsilon) d \|x\|_1^p \geq d \|x\|_p^p - \Theta(\epsilon) d \|x\|_p^p \\ &= (1 - \Theta(\epsilon)) d \|x\|_p^p \end{aligned}$$

where we used Lemma 9 in the third line. Altogether

$$\|u\|_p^p \geq (1 - \Theta(\epsilon))(1 - \Theta(\epsilon)) d \|x\|_p^p - \Theta(1)d\epsilon \|x\|_p^p = (1 - \Theta(\epsilon)) d \|x\|_p^p$$

□

**Lemma 14.** *We have  $\|u\|_p^p \leq (1 + \Theta(\epsilon)) d \|x\|_p^p$ .*

*Proof.* Define the set  $T = \{j : |u_j''| \leq |u_j''|\}$ . Decompose  $\|u\|_p^p$  into a sum

$$\sum_{j \in T} |u_j|^p + \sum_{j \notin T} |u_j|^p = T_1 + T_2$$

By Claim 11 we know that

$$T_1 = \sum_{j \in T} |u_j|^p = \sum_{j \in T} |u_j' + u_j''|^p \leq \sum_{j \in T} |u_j'|^p + \sum_{j \in T} p|2u_j'|^{p-1}|u_j''|$$

which, as in the proof of Lemma 13, can be bounded from above by

$$\sum_{j \in T} |u_j'|^p (1 + \Theta(\epsilon)) + \Theta(\epsilon) d \|x\|_p^p$$

At the same time, using Claim 12

$$T_2 = \sum_{j \notin T} |u_j|^p \leq \sum_{j \notin T} |2u_j''|^p = \Theta(\epsilon) d \|x\|_p^p$$

The lemma follows. □

Combining Lemma 14 and Lemma 13 yields the proof of the theorem. □

The above theorem shows that one can construct RIP- $p$  matrices for  $p \approx 1$  from good unbalanced expanders. In following, we show that this is not an accident: any binary matrix  $\Phi$  in which satisfies RIP-1 property with proper parameters, and with each column having exactly  $d$  ones, must be an adjacency matrix of a good unbalanced expander. This reason behind this is that if some set of vertices does *not* expand too well, then there are many collisions between the edges going out of that set. If the signs of the coordinates “following” those edges are different, the coordinates will cancel each other out, and thus the  $\ell_1$  norm of a vector will not be preserved.

The assumption that each column has exactly  $d$  ones is not crucial, since the RIP-1 property itself implies that the number of ones in each column can differ by at most factor of  $1 + \delta$ . All proofs in this paper are resilient to this slight unbalance. However, we decided to keep this assumption for the ease of notation.

**Theorem 2** *Consider any  $m \times n$  binary matrix  $\Phi$  such that each column has exactly  $d$  ones. If for some scaling factor  $S > 0$  the matrix  $S\Phi$  satisfies the  $\text{RIP}_{1,s,\delta}$  property, then the matrix  $\Phi$  is an adjacency matrix of an  $(s, \epsilon)$ -unbalanced expander, for*

$$\epsilon = \left(1 - \frac{1}{1+\delta}\right) / (2 - \sqrt{2}).$$

Note that, for small values of  $\delta > 0$ , we have  $\left(1 - \frac{1}{1+\delta}\right) / (2 - \sqrt{2}) \approx \delta / (2 - \sqrt{2})$ .

*Proof.* Let  $G = (A, B, E)$  be the graph with adjacency matrix  $\Phi$ . Assume that there exists  $X \subset A$ ,  $|X| = k' \leq k$  such that  $|N(X)| < dk'(1 - \epsilon)$ . We will construct two  $n$ -dimensional vectors  $y, z$  such that  $\|y\|_1 = \|z\|_1 = k'$ , but  $\|\Phi z\|_1 / \|\Phi y\|_1 \leq 1 - \epsilon(2 - \sqrt{2})$ , which is a contradiction.

The vector  $y$  is simply the characteristic vector of the set  $X$ . Clearly, we have  $\|y\|_1 = k'$  and  $\|\Phi y\|_1 = dk'$ .

The vector  $z$  is defined via a random process. For  $i \in X$ , define  $r_i$  to be i.i.d. random variables uniformly distributed over  $\{-1, 1\}$ . We define  $z_i = r_i$  if  $i \in X$ , and  $z_i = 0$  otherwise. Note that  $\|z\|_1 = \|y\|_1 = k'$ .

Let  $C \subset N(X)$  be the ‘‘collision set’’, i.e., the set of all  $j \in N(X)$  such that the number  $u_j$  of the edges from  $j$  to  $X$  is at least 2. Let  $|C| = l$ . By the definition of the set  $C$  we have  $\sum_j u_j \geq 2l$ . Moreover, from the assumption it follows that  $\sum_j u_j \geq 2\epsilon dk'$ .

Let  $v = \Phi z$ . We split  $v$  into  $v_C$  and  $v_{C^c}$ . Clearly,  $\|v_{C^c}\|_1 = k'd - \sum_j u_j$ . It suffices to show that  $\|v_C\|_1$  is significantly smaller than  $\sum_j u_j$  for *some*  $z$ .

**Claim 15.** *The expected value of  $\|v_C\|_2^2$  is equal to  $\sum_j u_j$ .*

*Proof.* For each  $j \in C$ , the coordinate  $v_j$  is a sum of  $u_j$  independent random variables uniformly distributed over  $\{-1, 1\}$ . The claim follows by elementary analysis.  $\square$

By Claim 15 we know that there *exists*  $z$  such that  $\|v_C\|_2 \leq \sqrt{\sum_j u_j} \leq \frac{\sum_j u_j}{\sqrt{2l}}$ . This implies that  $\|v_C\|_1 \leq \sqrt{l} \|v_C\|_2 \leq \frac{\sum_j u_j}{\sqrt{2}}$ . Therefore

$$\begin{aligned} \|v\|_1 &\leq \|v_C\|_1 + \|v_{C^c}\|_1 \leq \frac{\sum_j u_j}{\sqrt{2}} + dk' - \sum_j u_j = dk' - (1 - 1/\sqrt{2}) \sum_j u_j \\ &\leq dk' - (1 - 1/\sqrt{2}) \cdot 2\epsilon dk' = dk'[1 - \epsilon(2 - \sqrt{2})] \end{aligned}$$

$\square$

### 3. LP DECODING

In this section we show that if  $A$  is an adjacency matrix of an expander graph, then the LP decoding procedure can be used for recovering sparse approximations.

Let  $\Phi$  be an  $m \times n$  adjacency matrix of an unbalanced  $(2k, \epsilon)$ -expander  $G$  with left degree  $d$ . Let  $\alpha(\epsilon) = (2\epsilon)/(1 - 2\epsilon)$ . We also define  $E(X : Y) = E \cap (X \times Y)$  to be the set of edges between the sets  $X$  and  $Y$ .

**3.1. L1 Uncertainty Principle.** In this section we show that any vector from the kernel of an adjacency matrix  $\Phi$  of an expander graph is “smooth”, i.e., the  $\ell_1$  norm of the vector cannot be concentrated on a small subset of its coordinates. An analogous result for RIP-2 matrices and with respect to the  $\ell_2$  norm has been used before (e.g., in [KT07]) to show guarantees for LP-based recovery procedures.

**Lemma 16.** *Consider any  $y \in \mathbb{R}^n$  such that  $\Phi y = 0$ , and let  $S$  be any set of  $k$  coordinates of  $y$ . Then we have*

$$\|y_S\|_1 \leq \alpha(\epsilon)\|y\|_1.$$

*Proof.* Without loss of generality, we can assume that  $S$  consists of the largest (in magnitude) coefficients of  $y$ . We partition coordinates into sets  $S_0, S_1, S_2, \dots, S_t$ , such that (i) the coordinates in the set  $S_l$  are not larger (in magnitude) than the coordinates in the set  $S_{l-1}$ ,  $l \geq 1$ , and (ii) all sets but  $S_t$  have size  $k$ . Therefore,  $S_0 = S$ . Let  $\Phi'$  be a submatrix of  $\Phi$  containing rows from  $N(S)$ .

From the equivalence of expansion and RIP-1 property we know that  $\|\Phi' y_S\|_1 = \|\Phi y_S\|_1 \geq d(1 - 2\epsilon)\|y_S\|_1$ . At the same time, we know that  $\|\Phi' y\|_1 = 0$ . Therefore

$$\begin{aligned} 0 = \|\Phi' y\|_1 &\geq \|\Phi' y_S\|_1 - \sum_{l \geq 1} \sum_{(i,j) \in E, i \in S_l, j \in N(S)} |y_i| \\ &\geq d(1 - 2\epsilon)\|y_S\|_1 - \sum_{l \geq 1} |E(S_l : N(S))| \min_{i \in S_{l-1}} |y_i| \\ &\geq d(1 - 2\epsilon)\|y_S\|_1 - \frac{1}{k} \sum_{l \geq 1} |E(S_l : N(S))| \cdot \|y_{S_{l-1}}\|_1 \end{aligned}$$

From the expansion properties of  $G$  it follows that, for  $l \geq 1$ , we have  $|N(S \cup S_l)| \geq d(1 - \epsilon)|S \cup S_l|$ . It follows that at most  $d\epsilon 2k$  edges can cross from  $S_l$  to  $N(S)$ , and therefore

$$\begin{aligned} 0 &\geq d(1 - 2\epsilon)\|y_S\|_1 - \frac{1}{k} \sum_{l \geq 1} |E(S_l : N(S))| \cdot \|y_{S_{l-1}}\|_1 \\ &\geq d(1 - 2\epsilon)\|y_S\|_1 - d\epsilon 2 \sum_{l \geq 1} \|y_{S_{l-1}}\|_1 / k \\ &\geq d(1 - 2\epsilon)\|y_S\|_1 - 2d\epsilon\|y\|_1 \end{aligned}$$

It follows that  $d(1 - 2\epsilon)\|y_S\|_1 \leq 2d\epsilon\|y\|_1$ , and thus  $\|y_S\|_1 \leq (2\epsilon)/(1 - 2\epsilon)\|y\|_1$ .  $\square$

**3.2. LP recovery.** The following theorem provides recovery guarantees for the program  $P_1$ , by setting  $u = x$  and  $v = x_*$ .

**Theorem 3** *Consider any two vectors  $u, v$ , such that for  $y = v - u$  we have  $\Phi y = 0$ , and  $\|v\|_1 \leq \|u\|_1$ . Let  $S$  be the set of  $k$  largest (in magnitude) coefficients of  $u$ , then*

$$\|v - u\|_1 \leq 2/(1 - 2\alpha(\epsilon)) \cdot \|u - u_S\|_1$$

*Proof.* We have

$$\begin{aligned} \|u\|_1 \geq \|v\|_1 &= \|(u + y)_S\|_1 + \|(u + y)_{S^c}\|_1 \\ &\geq \|u_S\|_1 - \|y_S\|_1 + \|y_{S^c}\|_1 - \|u_{S^c}\|_1 \\ &= \|u\|_1 - 2\|u_{S^c}\|_1 + \|y\|_1 - 2\|y_S\|_1 \\ &\geq \|u\|_1 - 2\|u_{S^c}\|_1 + (1 - 2\alpha(\epsilon))\|y\|_1 \end{aligned}$$

where we used Lemma 16 in the last line. It follows that

$$2\|u_{S^c}\|_1 \geq (1 - 2\alpha(\epsilon))\|y\|_1$$

□

**Theorem 17.** Consider any two vectors  $u, v$ , such that for  $y = v - u$  we have  $\|\Phi y\|_1 = \beta \geq 0$ , and  $\|v\|_1 \leq \|u\|_1$ . Let  $S$  be the set of  $k$  largest (in magnitude) coefficients of  $u$ . Then

$$\|v - u_S\|_1 \leq 2/(1 - 2\alpha(\epsilon)) \cdot \|u_{S^c}\|_1 + \frac{2\beta}{d(1 - 2\epsilon)(1 - 2\alpha)}$$

*Proof.* Analogous to the proof of Theorem 3. □

#### 4. EXPERIMENTAL RESULTS

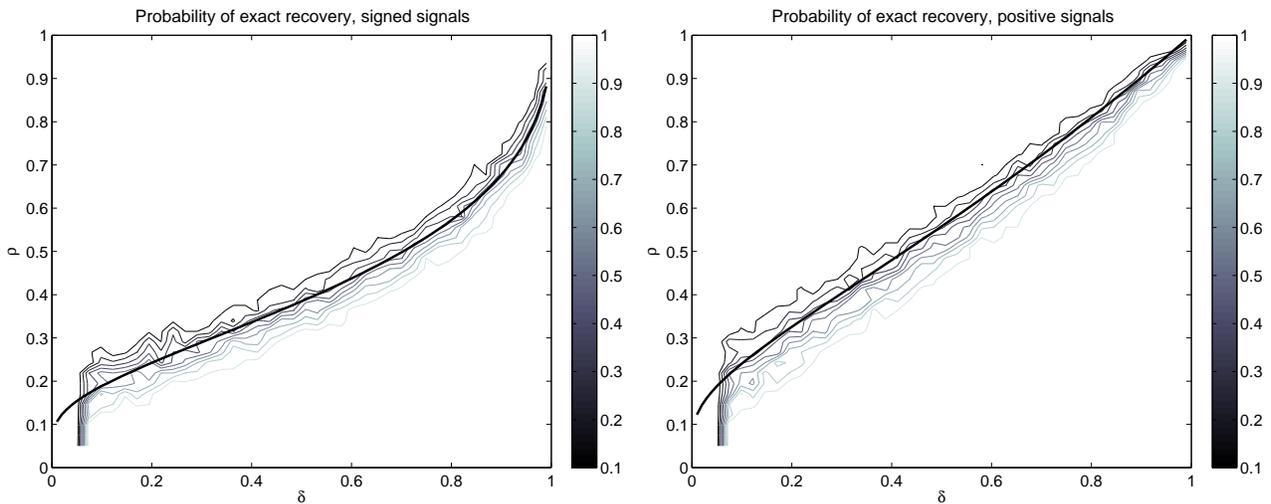


FIGURE 4. Probability of correct signal recovery of a random  $k$ -sparse signal  $x \in \{-1, 0, 1\}^n$  (left) and  $x \in \{0, 1\}^n$  (right) as a function of  $k = \rho m$  and  $m = \delta n$ , for  $n = 200$ . The probabilities were estimated by partitioning the domain into  $40 \times 40$  data points and performing 50 independent trials for each data point, using random sparse matrices with  $d = 8$ . The thick curve demarcates a phase transition in the ability of LP decoding to find the sparsest solution to  $Gx_* = Gx$  for  $G$  a Gaussian random matrix (see [DT06]). The empirical behavior for binary sparse matrices is consistent with the analytic behavior for Gaussian random matrices.

Our theoretical analysis shows that, up to constant factors, our scheme achieves the best known bounds for sparse approximate recovery. In order to determine the exact values of those constant factors, we show, in Figure 4, the empirical probability of correct recovery of a random  $k$ -sparse signal of dimension  $n = 200$  as a function of  $k = \rho m$  and  $m = \delta n$ . As one can verify, the empirical  $O(\cdot)$  constants involved are quite low. The thick curve shows the analytic computation of the phase transition between the survival of typical  $l$ -faces of the cross-polytope (left) and the polytope (right) under projection by  $G$  a Gaussian random matrix. This line is equivalent to a phase transition in the ability of LP decoding to find the sparsest solution to  $Gx_* = Gx$ , and, in effect, is representative of the performance of Gaussian matrices in this framework (see [Don06b] and [DT06] for more details). Gaussian measurement matrices with  $m = \delta n$  rows and  $n$  columns can recover signals with sparsity  $k = \rho m$  below the thick curve and cannot recover signals with sparsity  $k$  above the curve. This figure thus shows that the empirical behavior of binary sparse matrices with LP decoding is consistent with the analytic performance of Gaussian random matrices. Furthermore,

the empirical values of the asymptotic constants seem to agree. See [BI08] for further experimental data.

## 5. SUBLINEAR-TIME DECODING OF SPARSE VECTORS

In this section we focus on the recovery problem for the case where the signal vector  $x$  is  $k$ -sparse. The algorithm given here is essentially subsumed by the HHS algorithm provided in the Appendix (modulo the slightly higher reconstruction time and the number of measurements of the latter). The advantage of the algorithm from this section, however, is that it is very simple to present as well as analyze. This will enable us to illustrate the basic concepts without getting into technical details needed for the case of general signals  $x$ .

The algorithm we present here is essentially a simplification of the algorithm given in [Ind08]. The main difference is the assumption about the measurement matrix  $\Phi$ . In [Ind08] the matrix  $\Phi$  was assumed to be an “augmented” version of the adjacency matrix of an *extractor*. Here, we assume that is an “augmented” version of the adjacency matrix of an unbalanced expander (or alternatively, by Theorem 2, any binary matrix that satisfies an RIP-1 property). The latter assumption turns out to be the “right” one, resulting in further simplification of the algorithm.

To define the matrices, we start with a straightforward definition. If  $q$  and  $r$  are 0–1 vectors, we can view them as masks that determine which entries of a signal appear and which are zeroed out. For example, the signal  $q \circ x$  consists of the entries of  $x$  restricted to the nonzero components of  $q$ . The notation  $\circ$  indicates the componentwise or Hadamard product of two vectors. Given 0–1 matrices  $Q$  and  $R$ , we can form a matrix that encodes sequential restrictions by all pairs of their rows. We express this matrix as the *row tensor product*  $Q \otimes_r R$ .

The construction of the measurement matrix is as follows. Let  $G = (A, B, E)$  be a  $(k, \epsilon)$ -unbalanced expander,  $\epsilon = 1/8$ , with left degree  $d$ ,  $|A| = n$ ,  $|B| = m$ . Without loss of generality we can assume that  $m$  is a power of 2. Let  $\Psi$  be the adjacency matrix of  $G$ . The measurement matrix  $\Phi = \Psi \otimes_r B$  has  $m \log n$  rows. The matrix  $B$  is the *bit-test matrix*; its  $t$ th column contains the binary expansion of  $t$ . That is, For  $t = 0 \dots \log n - 1$  and  $j = 0 \dots m - 1$ , let  $l = t + j \log n$ . The  $(l + 1)$ -th row  $\Phi_{l+1}$  of  $\Phi$  is such that, for any  $i = 0 \dots m - 1$ ,  $(\Phi_{l+1})_{i+1} = (\Psi)_{j+1} \cdot \text{bin}_t(i)$ , where  $\text{bin}_t(i)$  is the  $t$ -th least significant bit in the binary representation of  $i$ .

The purpose of the above augmentation is as follows. Consider a specific  $k$ -sparse vector  $x$ . Let  $\text{supp}(x)$  denote the support of  $x$ , i.e., the set of indices of the non-zero coordinates of  $x$ . Assume that the  $(j + 1)$ -th row  $r$  of the matrix  $\Psi$  is such that  $|\text{supp}(x) \cap \text{supp}(r)| = 1$ , and let  $i + 1 \in \text{supp}(x) \cap \text{supp}(r)$ . Then, from the values of  $\Phi_{j \log n + 1} x, \dots, \Phi_{j \log n + \log n} x$ , we can recover both  $i$  and  $x_{i+1}$ . This is because for each of those rows, the value of  $\Phi_{j \log n + t + 1} x$  is equal to 0 if  $\text{bin}_t(i) = 0$ , or  $x_{i+1}$  otherwise; moreover, at least one of the values is non-zero. Of course, if  $|\text{supp}(x) \cap \text{supp}(r)| \neq 1$ , then the algorithm might “recover” an incorrect pair  $(i, \text{val})$ . This can be detected if  $\text{val} = 0$ ; otherwise, the error might be undetected.

**Theorem 18.** *Let  $\Phi$  be a matrix described above. Then for any  $k$ -sparse  $x$ , given  $\Phi x$ , we can recover  $x$  in time  $O(m \log^2 n)$ .*

*Proof.* The main component of the recovery algorithm is the procedure  $\text{Reduce}(\Phi x)$ , which returns a vector  $y$  such that  $\|x - y\|_0 \leq \|x\|_0 / 2$ . The procedure maintains a vector  $\text{votes}[\cdot]$  of multisets. Initially, all entries are set to  $\emptyset$ .

```

Reduce( $\Phi x$ ):

  for  $j = 1 \dots m$ 
    compute  $(l, val)$  such that if  $\text{supp}(\Phi)_j \cap \text{supp}(x) = \{i\}$  then  $x_i = val$ 
    if  $val \neq 0$  then  $\text{votes}[l] = \text{votes}[l] \cup \{val\}$ 
   $y = 0$ 
  for  $i = 1 \dots n$  such that  $\text{votes}[i] \neq \emptyset$ 
    if  $\text{votes}[i]$  contains  $\geq d/2$  copies of  $val$  then  $y_i = val$ 
  return  $y$ 

```

**Claim 19.** For any  $k$ -sparse vector  $x$ , the procedure REDUCE returns a vector  $y$  such that  $x - y$  is  $k/2$ -sparse.

*Proof.* From the expanding properties of  $G$ , it follows that at most  $\epsilon d \|x\|_0$  rows of  $\Psi$  return an incorrect pair  $(i, val)$ . Each set of  $d/2$  incorrect pairs can change the outcome for at most 2 positions of  $y$ . Thus, at most  $4\epsilon \|x\|_0 = \|x\|_0/2$  positions of  $y$  can be incorrect.  $\square$

The final algorithm invokes RECOVER( $\Phi x$ ).

```

Recover( $\Phi x$ )
  if  $\Phi x = 0$  then return 0
   $y = \text{Reduce}(\Phi x)$ 
  { We now need to recover  $x - y$  }
   $z = \text{Recover}(\Phi(x - y))$ 
  return  $y + z$ 

```

$\square$

## 6. CONCLUSION

We show in this paper that the geometric and the combinatorial approaches to sparse signal recovery are different manifestations of a common underlying phenomenon. Thus, we are able to show a unified perspective on both approaches—the key unifying elements are the adjacency matrices of unbalanced expanders.

In most of the recent applications of *compressed sensing*, a physical device instantiates the measurement of  $x$  and, as such, these applications need measurement matrices which are conducive to physical measurement processes. This paper shows that there is another, quite different, large, natural class of measurement matrices, combined with the same (or similar) recovery algorithms for sparse signal approximation. These measurement matrices may or may not be conducive to physical measurement processes but they are quite amenable to computational or digital signal measurement. Our work suggests a number of applications in digital or computational “sensing” such as efficient numerical linear algebra and network coding.

The preliminary experimental analysis exhibits interesting high-dimensional geometric phenomena as well. Our results suggest that the projection of polytopes under Gaussian random matrices is similar to that of projection by sparse random matrices, despite the fact that Gaussian random matrices are quite different from sparse ones.

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## APPENDIX A. HHS( $p$ ) DECODING

In this section we focus on the general recovery problem for an arbitrary vector  $x$ . We show that, as in the previous algorithm in Section 5, we can use the adjacency matrices of unbalanced expanders, suitably augmented and concatenated, to obtain an explicit construction of matrices that are designed for the sub-linear time combinatorial recovery algorithm HHS in [GSTV07]. We call this modified algorithm the HHS( $p$ ) algorithm. For the sake of exposition, we highlight the necessary changes in the construction of the matrices and the algorithm and summarize the remaining, unchanged portions of the algorithm. Similarly, for the analysis of HHS( $p$ ), we present only those portions which are significantly different from the original analysis and summarize those which are not. We establish the following result.

**Theorem 4** *Let  $x \in \mathbb{R}^n$  and fix a sparsity parameter  $k$  and a number  $\epsilon \in (0, 1)$ . There is a measurement matrix  $\Psi$ , which we can construct explicitly or randomly, with the following property. The HHS( $p$ ) algorithm, given measurements  $v = \Psi x$  of  $x$ , returns an approximation  $\hat{x}$  of  $x$  with  $O(k/\epsilon)$  nonzero entries. The approximation satisfies*

$$\|x - \hat{x}\|_p \leq \epsilon k^{1/p-1} \|x - x_k\|_1.$$

*Let  $R$  denote the size of the measurements for either an explicit or random construction. Then, the HHS( $p$ ) algorithm runs in time  $\text{poly}(R)$ .*

For explicit constructions  $R = O(k2^{\log \log n^{O(1)}})$  and for random constructions  $R = O(k \text{ polylog } n)$ .

**Corollary 20.** *If we truncate the output  $\hat{x}$  of the algorithm to the  $k$  largest terms, then the modified output  $\hat{x}_k$  satisfies*

$$\begin{aligned} \|x - \hat{x}_k\|_p &\leq \|x - x_k\|_p + 2\epsilon k^{1/p-1} \|x - x_k\|_1 \quad \text{and} \\ \|x - \hat{x}_k\|_1 &\leq (1 + 3\epsilon) \|x - x_k\|_1. \end{aligned}$$

**A.1. The measurement matrices.** This subsection describes how to construct the measurement matrix  $\Psi$ . We note that this construction has the same super-structure as the random construction in [GSTV07] and, as such, we highlight the necessary changes while summarizing the similar pieces. For ease of exposition, we set  $\epsilon = 1$ . The matrix  $\Psi$  consists of two pieces: an identification matrix  $\Omega$  and an estimation matrix  $\Phi$ . We use the first part of the sketch to identify indices of significant components of the signal quickly and then we use the second part to estimate the coefficients of those terms.

We use concatenated copies of the (normalized) adjacency matrices  $\Gamma$  of  $(k', \epsilon')$ -unbalanced expanders with left degree  $d$ . In what follows, we let the sparsity parameter  $k'$  vary but fix  $p = 1 + 1/\log(n)$ . Normalize by the factor  $d^{-1/p}$ . By Theorem 1, such matrices satisfy the

RIP $_{p,k',\delta}$  property for  $p = 1 + 1/\log n$  and  $\delta \leq C\epsilon'$ . Fix  $\epsilon'$  so that  $4\epsilon' + 1/(2d^{1/p}) < 1/2$ . Finally, we note that the number of rows in such a matrix is  $k'2^{(\log \log n)^{O(1)}}$  for known *explicit* constructions of such matrices and is  $k'$  polylog( $n$ ) for random constructions. To simplify our accounting below, we denote the number of rows by  $R$  for either the explicit or the random constructions.

**A.1.1. Identification matrix.** The identification matrix  $\mathbf{\Omega}$  is a 0–1 matrix with dimensions  $O(R \text{ polylog}(n)) \times n$ . It consists of a combination of a three structured matrices. Formally,  $\mathbf{\Omega}$  is the row tensor product  $\mathbf{\Omega} = \mathbf{B} \otimes_r \mathbf{A}$ . The *bit-test matrix*  $\mathbf{B}$  has dimensions  $O(\log n) \times n$ , and the *isolation matrix*  $\mathbf{A}$  has dimensions  $O(R \text{ polylog}(n)) \times n$ .

The isolation matrix  $\mathbf{A}$  is a 0–1 matrix with a hierarchical structure. It consists of  $\log(k)$  blocks  $\mathbf{A}^{(j)}$  labeled by  $j = 1, 2, 4, 8, \dots, J$ , where  $J = k$ . Each block, in turn, has further substructure as a row tensor product of a collection of 0–1 matrices:  $\mathbf{A}_{r,s}^{(j)} = \mathbf{R}_r^{(j)} \otimes_r \mathbf{S}_s^{(j)}$  for  $s = 1, 2, 4, \dots, k$  and  $r = 2s, 4s, 8s, \dots, n$ . The first matrix  $\mathbf{S}_s^{(j)}$  is called the *sifting matrix* and the second matrix  $\mathbf{R}_r^{(j)}$  is called the *noise reduction matrix*. Each sifting matrix  $\mathbf{S}_s^{(j)}$  is the normalized adjacency matrix of an  $(s, \epsilon')$ -unbalanced expander (or, equivalently, a normalized 0–1 RIP $_{p,s,\delta}$  matrix).

The purpose of the noise reduction matrix  $\mathbf{R}_r^{(j)}$  is to attenuate the noise in a signal that has a single large component. It is also a 0–1 valued matrix constructed as follows. For each pair of indices  $(r, s)$ , let  $\beta \approx r/s$  be a prime power corresponding to a finite field of size  $\beta$ . Then we set each noise reduction matrix to be the Nisan-Wigderson generator over the field of size  $\beta$ . That is, we index the rows of the matrix by pairs  $(a, b)$  of field elements and index columns by polynomials  $q$  of degree at most polylog( $n$ ). Position  $((a, b), q)$  is 1, if  $q(a) = b$ , and 0, otherwise. Such a construction produces a matrix with  $O(\beta^2)$  rows. See [DeV07] or [CM06] for similar constructions of similar sizes.

We note that the dimensions of the product of matrices  $\mathbf{A}_{r,s}^{(j)} = \mathbf{R}_r^{(j)} \otimes_r \mathbf{S}_s^{(j)}$  are the critical ones (not the dimensions of the individual matrices). Each block is of dimension  $O(R \text{ polylog}(n)) \times n$ .

**A.1.2. The estimation matrix.** The estimation matrix  $\mathbf{\Phi}$  is an RIP $_{p,K,\delta}$  matrix (or, equivalently, the adjacency matrix of an  $(K, \epsilon)$ -unbalanced expander). The parameter  $K = O(k \text{ polylog}(n))$  specifies how many spikes are identified during the identification stage. To ensure that the columns of  $\mathbf{\Phi}$  have unit  $\ell_p$  norm, we scale the matrix appropriately.

**A.2. The HHS( $p$ ) algorithm.** The structure of the HHS( $p$ ) algorithm remains unchanged. We include pseudo-code in Figure 5 for completeness. This algorithm is an iterative procedure with several steps per iteration, including identifying a small set of signal positions that contain a substantial fraction of the  $\ell_p$  weight, then estimating their values, and finally subtracting their encoded contribution from the initial measurements.

**A.3. Proof sketches.** The goal of the algorithm is to identify a small set of signal components that carry most of the  $p$ -energy in the signal and to estimate the coefficients of those components well. We argue that, when our signal estimate is poor, the algorithm makes substantial progress toward this goal. We focus on the analysis of the algorithm in the case when our signal estimate is poor as this is the critical case. More precisely, assume that the current approximation  $a$  satisfies

$$\|x - a\|_p > k^{1/p-1} \|x - x_k\|_1. \quad (\text{A.1})$$

Then one iteration produces a new approximation  $a^{\text{new}}$  for which  $\|x - a^{\text{new}}\|_p \leq \frac{1}{2} \|x - a\|_p$ . In this fashion, the algorithm improves the  $\ell_p$ -norm of the error geometrically until it is small enough. Contrast this estimate with that of the previous section (where we reduced the  $\ell_0$  norm of the error at each step). We will show this major result in a series of steps:

**Algorithm: HHS( $p$ ) Pursuit**

Inputs: The number  $k$  of spikes, the HHS measurement matrix  $\Psi$ ,  
the initial sketch  $v = \Psi x$ , and  $\Delta$  the dynamic range of  $x$

Output: A list  $L$  of  $O(k)$  spikes

For each iteration  $i = 0, 1, \dots, O(\log \Delta)$  {

For each scale  $j = 1, 2, 4, \dots, O(k)$  {

Initialize  $L' = \emptyset$ .

For each row of  $A^{(j)}$  {

Use the  $O(\log n)$  bit tests to identify one spike location

}

Retain a list  $L'_j$  of the spike locations  
that appear  $\Omega(dj)$  times each

Update  $L' \leftarrow L' \cup L'_j$

}

Estimate coefficients for the indices in  $L'$  by forming  $\Phi_{L', s_{\text{est}}}^\dagger$   
with Jacobi iteration

Update  $L$  by adding the spikes in  $L'$

If an index is duplicated, add the two values together

Prune  $L$  to retain the  $O(k)$  largest spikes

Encode these spikes with measurement matrix  $\Psi$

Subtract encoded spikes from original sketch  $v$  to form  
a new residual sketch  $s$

}

FIGURE 5. Pseudocode for the HHS( $p$ ) Pursuit algorithm

- (1) First, we obtain a fundamental relationship between the large and the small entries in a signal in Lemmas 21, 22.
- (2) Next, we show in Lemma 23 that the sifting matrix isolates significant coefficients in a moderate amount of noise.
- (3) Lemma 27 proves that the noise reduction matrix reduces the  $\ell_1$ -norm of the noise enough so that we can accurately identify the indices of the significant coefficients.
- (4) Finally, we use a small  $\text{RIP}_{p,L,\delta}$  matrix to estimate the values of the coefficients in Lemma 28.

We describe some generic properties of signals that are important in the analysis. Given a signal  $g$ , we write  $g_t$  to denote the signal obtained by zeroing out all the components of  $g$  except the  $t$  largest in magnitude (break ties lexicographically). We refer to  $g_t$  as the *head* of the signal; it is the best approximation of the signal using at most  $t$  terms with respect to any monotonic norm (such as  $\ell_p$ ). The vector  $g - g_t$  is called the *tail* of the signal. First, we show that the  $\ell_p$  norm of the tail of a signal is much smaller than the  $\ell_1$  norm of the entire signal.

**Lemma 21.** *For any signal  $g$ , it holds that*

$$\|g - g_t\|_p \leq \left(\frac{1}{p-1}\right)^{1/p} t^{1/p-1} \|g\|_1.$$

*Proof.* Assume that the terms  $g(i)$  of  $g$  are arranged in decreasing order  $|g(1)| \geq |g(2)| \geq \dots \geq |g(n)|$ . We observe that each term  $g(i)$  in  $g$  satisfies  $|g(i)| \leq i^{-1} \|g\|_1$  and we can bound the  $\ell_p$  norm

of the tail as

$$\begin{aligned} \|g - g_t\|_p &\leq \left( \sum_{i=t+1}^n \frac{1^{-p}}{i} \|g\|_1^p \right)^{1/p} \\ &\leq \left( \|g\|_1^p \frac{1}{p-1} t^{1-p} \right)^{1/p} \\ &\leq \left( \frac{1}{p-1} \right)^{1/p} t^{1/p-1} \|g\|_1 \end{aligned}$$

□

We note that for  $p = 1 + \iota$ , the constant  $C_p = \left(\frac{1}{p-1}\right)^{1/p}$  is bounded below by  $1/\iota$  and from above by  $1/\iota^2$  so if  $p = 1 + 1/\log(n)$ , then  $\log(n) \leq C_p \leq \log^2(n)$ .

Let  $r = x - a$  denote the residual signal at the current approximation  $a$  and let  $\ell = O(k)$  be the number of terms in the approximation  $a$  at each iteration. When condition (A.1) holds, most of the  $p$ -energy in the signal is concentrated in its largest components. The next lemma relates the  $\ell_p$  norm of the residual to the  $\ell_p$  norm of its tail  $r - r_\ell$  and to the  $\ell_1$  norm of the entire residual.

**Lemma 22** (Head and Tail). *Suppose that (A.1) holds. Let  $\ell = O(k)$  be the number of terms in the approximation  $a$  at each iteration. Then the following bounds hold.*

$$\|r\|_p \geq C_p \|r - r_\ell\|_p \tag{A.2}$$

$$\|r\|_p \geq C \ell^{1/p-1} \|r\|_1. \tag{A.3}$$

*Proof.* Observe that because the approximation  $a$  contains no more than  $\ell$  terms in each iteration, the number of terms in the head of the residual  $r_\ell = a - x_k$  is not more than  $\ell + k = O(k)$ . Rather than keep careful track of the exact number of terms in each approximation and each residual, we will, by abuse of notation, suppress all constants and refer to the number of terms as  $O(k) = \ell$ . Therefore, (A.1) implies that

$$\|r\|_p = \|x - a\|_p > k^{1/p-1} \|x - x_k\|_1 = k^{1/p-1} \|(x - a) + (a - x_k)\|_1 = \ell^{1/p-1} \|r - r_\ell\|_1. \tag{A.4}$$

Now, apply Lemma 21 to the residual tail  $r - r_\ell$  with  $t = \ell$  to obtain

$$\|r - r_\ell\|_1 \geq C_p \ell^{1-1/p} \|r - r_\ell\|_p.$$

If we combine this relation with our previous calculation, we find

$$\|r\|_p \geq C_p \|r - r_\ell\|_p$$

which is our first desired inequality.

To prove the second inequality, we use the Cauchy-Schwarz inequality to see that

$$\|r\|_p \geq \|r_\ell\|_p \geq \ell^{1/p-1} \|r_\ell\|_1.$$

Finally, we add this inequality to (A.4) to obtain

$$\|r\|_p \geq C \ell^{1/p-1} \|r\|_1.$$

□

Now, we turn to the identification of significant signal entries. We pull these off in bands of decreasing magnitude. For exposition, we assume that  $\|g\|_1 = 1$ . We can think of the action of one submatrix  $\mathcal{S}_t^{(j)}$  as

$$\mathcal{S}_t^{(j)} : g \mapsto [h^1 \quad h^2 \quad \dots \quad h^N],$$

mapping each input signal to a collection of output signals.

**Lemma 23** (Isolation). *Let  $g$  be a signal and let  $I$  be a subset of  $\{1, 2, \dots, n\}$  with  $s \leq |I| < 2s$ . Let  $\ell = |I|$ . We apply the sifting operator  $\mathbf{S}_s^{(j)}$  to  $g$ . For at least  $(1 - \rho')\ell = (1 - (4\epsilon' + 1/(2d^{1/p})))$  of the indices  $i \in I$ , there is an output signal  $h$  of the form*

$$h = g_i e_i + \nu \quad \text{and} \quad \|\nu\|_1 \leq \frac{2}{\ell} \|g\|_1.$$

*Proof.* To prove this result, we must show a sequence of shorter results. Our goal is to isolate significant spikes from one another and to reduce the contribution of the rest of the signal to these isolations.

**Definition 24.** *An  $(s, r)$  band is a band of spikes of magnitude between  $1/r$  and  $1/(2r)$ . There is some  $s$  (a power of 2) such that the number of spikes in the band is between  $s$  and  $2s$ . If the  $p$ -contribution,  $sr^{-p}$ , of this band to the current residual error  $\|x - a\|_p^p$  is greater than the average band's contribution,  $\frac{1}{\log n} \left( k^{1/p-1} \|x - a\|_p \right)^p$ ; i.e.,*

$$sr^{-p} \geq \frac{1}{\log n} \left( k^{1/p-1} \|x - a\|_p \right)^p$$

*then we call this band significant.*

The next lemma tells us how many spikes lie above a significant band.

**Lemma 25.** *If an  $(s, r)$  band is significant, then there are at most  $s \text{ polylog}(n)$  terms of magnitude greater than  $1/r$  in the current residual.*

*Proof.* If the  $s$  spikes of magnitude  $1/r$  have  $p$ -energy at least as big as  $s'$  spikes of magnitude  $1/r' > 1/r$ , then

$$s(1/r)^p \geq s'(1/r')^p > s'(1/r)^p$$

and so the number of larger spikes must be less than  $s$ ,  $s' \leq s$ , as  $p > 1$ . Because there are only  $\log n$  possible  $s'$ , if we take a union over all  $s'$ , we achieve the desired result.  $\square$

**Definition 26.** *We say that a spike with index  $i$  is isolated from other spikes with indices in a set  $I$  by a matrix  $\mathbf{\Gamma}$  if there is a row in  $\mathbf{\Gamma}$  that has a one in column  $i$  and no other one in any column in  $I$ .*

We are ready to begin the proof of Lemma 23. Our goal is to show that  $\mathbf{\Gamma} = \mathbf{S}_s^{(j)}$  isolates from each other the vast majority of spikes of magnitude  $\geq 1/r$ , and, hence, isolates the majority of those with magnitude equal to  $1/r$  from the larger spikes.

By the expansion properties of  $\mathbf{\Gamma}$ , at least  $(1 - 4\epsilon')\ell$  distinct spikes in the original signal are isolated  $d/2$  times in the measurements. Next, we show that  $\mathbf{\Gamma}$  not only isolates spikes but also reduces the  $\ell_1$  norm of any noise, if it is present. We will argue that few of the good output signals have a lot of noise. Let  $u_m$  denote the magnitude of the  $m$ th output position. We observe that the  $1 \rightarrow 1$  operator norm of the matrix  $\mathbf{\Gamma}$  is  $d^{1-1/p}$ . By Markov's inequality,

$$\begin{aligned} \#\left\{m \mid u_m \geq \frac{2}{\ell} \|g\|_1\right\} &\leq \frac{\ell}{2\|g\|_1} \sum u_m \\ &= \frac{\ell}{2\|g\|_1} d^{1-1/p} \|g\|_1 \\ &= \frac{d^{1-1/p} \ell}{2}. \end{aligned}$$

Therefore, there are at least  $(1 - \rho)d\ell/2$  isolating rows and no more than  $\frac{d\ell}{2}d^{-1/p}$  of the rows are corrupted by noise with large  $\ell_1$  norm. This leaves  $d\ell/2((1 - 4\epsilon') - \frac{1}{2d^{1/p}}) = (1 - \rho')d\ell/2$  good rows with a single distinct spike and a small amount of noise. With judicious choice of unbalanced expander parameters, we have  $(1 - \rho')\ell$  isolated spikes corrupted by a small amount of noise.  $\square$

The isolation lemma 23 produces signals which consist of single isolated spikes plus some noise. In order to apply bit-testing accurately, we must reduce the  $\ell_1$  norm of the noise from a factor  $1/s$  of the original  $\ell_1$  norm by a further factor  $s/r$  (i.e., to a factor  $1/r$  of the original noise). Assume, for exposition, that the original  $\ell_1$  norm is one. To achieve this reduction, we use a noise reduction matrix  $\mathbf{R}_r^{(j)}$ .

**Lemma 27** (Noise reduction). *Let  $h = g_i e_i + \nu$  with  $\|\nu\|_1 \leq 1/s$ . We apply the noise reduction operator  $\mathbf{R} = \mathbf{R}_r^{(j)}$  to  $h$ . In at least half of the output signals, we have signals of the form*

$$\delta + \nu \quad \text{with} \quad \|\nu\|_1 \leq C/r.$$

*Proof.* Consider the submatrix  $\mathbf{R}'$  of  $\mathbf{R}$  obtained by extracting the  $\beta$  rows of  $\mathbf{R}$  that have a 1 in column  $i$  and then removing column  $i$  itself. By construction,  $\mathbf{R}'$  has at most  $\text{polylog}(n)$  ones in any column, so its 1-to-1 operator norm is at most  $\text{polylog}(n)$ . Thus the average over rows of  $\mathbf{R}'$  of  $\|\mathbf{R}'\nu\|_1$  is at most  $\frac{\text{polylog}(n)}{\beta s} \approx 1/r \text{ polylog}(n)$ , or at most  $1/2r$  if we incorporate extra log factors into  $\beta$ .

In addition, the number of rows in the matrix  $\mathbf{A}_{r,s}^{(j)}$  is  $s \cdot \max(1, (r/s)^2)$ , where  $s \leq k$  and  $sr^{-p} \geq k^{p-1}$ . If  $s/r \geq 1$ , the number of rows is  $s \leq k$ . If  $s/r \leq 1$ , the number of rows is  $r^2/s$ . Observe that  $sr^{-p} \geq k^{1-p}$  and  $r/s \geq 1$  imply  $1/r \geq 1/k$ . Since  $sr^{-p} \geq k^{1-p}$ , it follows that

$$\frac{r^2}{s} = \frac{r^p}{sr^{p-2}} \leq \frac{k^{p-1}}{r^{p-2}} \leq k^{2-p} k^{p-1} = k.$$

$\square$

The estimation portion of the algorithm is exactly the same as the HHS algorithm. Let  $L$  be the set of identified candidates,  $|L| \leq K$ . By the RIP- $p$  property, the submatrix of  $\Phi$  given by columns indexed by  $L$  is invertible and the inverse  $\Phi_L^+$  is bounded in the appropriate operator norm. we can apply  $\Phi_L^+$  to  $\Phi x$  by Jacobi iteration in time  $O(K^2 \text{polylog}(n))$  to estimate the coefficients in  $L$ . In the exact case of  $x_{\bar{L}} = 0$ , we recover  $x$  exactly. Otherwise, the RIP- $p$  property assures that the  $p$ -norm of the vector of estimates is bounded in terms of the  $p$ -norm of  $x - x_K$ ; i.e., this estimation process introduces errors that are small compared with the error associated with missing the  $n - K$  other terms in  $x$ . Its proof is a small modification of the proof in [GSTV07], which itself was originally proven by Rudelson.

**Lemma 28.** *Suppose that  $\Phi$  is an  $m \times n$  RIP( $p, K, \delta$ ) matrix. Then the following holds for every vector  $x \in \mathbb{R}^n$ .*

$$\|\Phi x\|_p \leq (1 + \delta) \left( \|x\|_p + K^{1/p-1} \|x\|_1 \right).$$

For completeness, we conclude this section by showing that a mixed norm error bound of the form  $\ell_p \leq C/k^{1-1/p} \ell_1$  gives us an  $\ell_1$  error bound  $\ell_1 \leq C\ell_p$ .

**Theorem 29.** *Suppose*

$$\|\tilde{x} - x\|_p \leq \epsilon k^{1/p-1} \|x_k - x\|_1.$$

*Then  $\|\tilde{x}_k - x\|_p \leq \|x_k - x\|_p + 2\epsilon k^{1/p-1} \|x_k - x\|_1$  and  $\|\tilde{x}_k - x\|_1 \leq (1 + 3\epsilon) \|x_k - x\|_1$ .*

*Proof.* Let  $H$  denote  $\text{supp}(x_k) \cup \text{supp}(\tilde{x}_k)$ , so that  $|H| \leq 2k$ . Let  $H^c$  denote the complement of  $H$ .

$$\begin{aligned}
\|\tilde{x}_k - x\|_p &\leq \|(\tilde{x}_k - x)|_H\|_p + \|(\tilde{x}_k - x)|_{H^c}\|_p \\
&\leq \|(\tilde{x}_k - \tilde{x})|_H\|_p + \|(\tilde{x} - x)|_H\|_p + \|(\tilde{x}_k - x)|_{H^c}\|_p \\
&\leq \|(x_k - \tilde{x})|_H\|_p + \|(\tilde{x} - x)|_H\|_p + \|(\tilde{x}_k - x)|_{H^c}\|_p \\
&\leq \|x_k - x\|_p + 2\|(\tilde{x} - x)|_H\|_p \\
&\leq \|x_k - x\|_p + 2\|\tilde{x} - x\|_p \\
&\leq \|x_k - x\|_p + 2\epsilon k^{1/p-1} \|x_k - x\|_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\tilde{x}_k - x\|_1 &= \|(\tilde{x}_k - x)|_H\|_1 + \|(\tilde{x}_k - x)|_{H^c}\|_1 \\
&\leq \|(\tilde{x}_k - \tilde{x})|_H\|_1 + \|(\tilde{x} - x)|_H\|_1 + \|(\tilde{x}_k - x)|_{H^c}\|_1 \\
&\leq \|(x_k - \tilde{x})|_H\|_1 + \|(\tilde{x} - x)|_H\|_1 + \|(\tilde{x}_k - x)|_{H^c}\|_1 \\
&\leq \|x_k - x\|_1 + 2\|(\tilde{x} - x)|_H\|_1 \\
&\leq \|x_k - x\|_1 + 2(2k)^{1-1/p} \|(\tilde{x} - x)|_H\|_p \\
&\leq \|x_k - x\|_1 + 2(2k)^{1-1/p} \|\tilde{x} - x\|_p \\
&\leq \|x_k - x\|_1 + 2^{2-1/p} \epsilon \|x - x_k\|_1 \\
&\leq (1 + 3\epsilon) \|x_k - x\|_1.
\end{aligned}$$

□