ECE 553, Spring 2008 **Problem Set #6 Solution**

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Solutions:

1. Introduce ν as the Lagrange multiplier to the terminal state constraint $x_1^2(t_f) + x_2^2(t_f) = 1$. The Hamiltonian is $H = \frac{1}{2}\alpha^2 x_1^2 + \frac{1}{2}x_2^2 + p_1(x_2) - p_2(x_2 + x_1 u)$ and it gives the costate equations:

$$\begin{cases} \dot{p}_1 &= -\alpha^2 x_1 + p_2 u, \quad p_1(t_f) = \nu x_1(t_f) \\ \dot{p}_2 &= x_2 - p_1 + p_2 u, \quad p_2(t_f) = \nu x_2(t_f). \end{cases}$$

   There exists a singular control if $H_u = -p_2 x_1 = 0$ on some nonzero interval.

   Now x_1 cannot be zero since this will make x_2 zero, and the system can never be made to satisfy the terminal state constraint. Hence, we conclude that on the singular arc $p_2 \equiv 0$. This implies $\dot{p}_2 = 0 \Rightarrow p_1 = -x_2 \Rightarrow \dot{p}_1 = -\dot{x}_2 \Rightarrow -\alpha^2 x_1 = x_2 + x_1 u$. The optimal control on the singular arc is

$$u^* = -\frac{\alpha^2 x_1 + x_2}{x_1}.$$

   Since the terminal time t_f is free, $H|_{t_f} = 0$ implies $\frac{1}{2}\alpha^2 x_1^2(t_f) + \frac{1}{2}x_2^2(t_f) - x_2^2(t_f) = 0$. This yields

$$(\alpha^2 + 1)x_1(t_f)^2 = 1.$$

   Furthermore, by the time-invariance of $H \equiv 0$, on the singular arc, we have

$$\frac{1}{2}\alpha^2 x_1^2(t) = \frac{1}{2}x_2^2(t).$$

   Hence $x_2(t) = \pm\alpha x_1(t)$.

2. The HJB is

$$-\frac{\partial V}{\partial t} = \min_u \left\{ \frac{\partial V}{\partial x} x u + (xu)^2 \right\}, \quad V(1, x) = x^2.$$

   The minimizing control is

$$\frac{\partial V}{\partial x} x + 2x^2 u = 0 \quad \Rightarrow \quad u = -\frac{1}{2x}\frac{\partial V}{\partial x},$$

   where we have assumed that $x \neq 0$ (which indeed can be shown to be the case). Then the HJB becomes:

$$-\frac{\partial V}{\partial t} = -\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2, \quad V(1, x) = x^2.$$

   Consider the quadratic solution $V(t, x) = p(t)x^2$, and use this in the HJB:

$$-\dot{p}x^2 = -p^2 x^2; \quad p(1) = 1.$$

   This yields $\dot{p} = p^2$ and therefore

$$p(t) = \frac{1}{2 - t}.$$

   Hence

$$u^*(t) = -\frac{1}{2x}\frac{\partial V}{\partial x} = -\frac{1}{(2 - t)}, \quad 0 \le t \le 1.$$

   You should use this control to verify that the resulting $x^*(t)$ is never zero, as assumed above.

3. For $\alpha < 0$, A is stable, hence (A, B) is stabilizable, and (A, C) is also detectable for all β, ρ.

   For $\alpha \leq 0$, (A, B) is stabilizable if and only if $\beta \neq 0$; (A, C) is detectable if and only if $\rho + \alpha + 1 \neq 0$.

   (a) From above, either $\alpha < 0$, or $\alpha \geq 0$ and $\beta \neq 0, \rho + \alpha + 1 \neq 0$.

   (b) For the solution of ARE to be positive definite, we further need observability. The requires $\rho \neq 0$ in addition to the above conditions.

   (c) For this we need the stabilizability and no unobservable eigenvalues of A on the imaginary axis. From this we obtain either $\alpha < 0$, or $\alpha = 0, \rho \neq -1, \beta \neq 0$, or $\alpha > 0, \beta \neq 0$.

4. Let $v = u + x_1$, and view it as the new control. In terms of v we have

$$\begin{aligned} J(v) &= \int_0^\infty (|x|_{\tilde{Q}}^2 + v^2)\, dt, \\ \dot{x} &= \tilde{A}x + Bv, \end{aligned}$$

   where

$$\tilde{Q} = \begin{bmatrix} \rho^2 - 1 & \rho \\ \rho & 2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -2 & 1 \\ \beta & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \beta \end{bmatrix}.$$

   Similar as the previous problem, we obtain:

   (a) $\rho^2 \geq 2$ and either $\alpha < 0$ or $\alpha \geq 0$ and $\beta \neq 0$.

   (b) $\rho^2 \geq 2, 1 + 2\sqrt{2} + \alpha\sqrt{2} - 2\beta \neq 0$, and either $\alpha < 0$ or $\alpha \geq 0$ and $\beta \neq 0$.

   (c) Same as part (a).

Please report to yima@uiuc.edu if you find any typos.