ECE 553, Spring 2008 **Problem Set #5 Solution**

Posted: April 15th, 2008

Solutions:

1. The eigenvalues of the system matrix A are -1 and -2 which are real. There can be at most 1 switch. The switching curve is a curve in 2-D space, which will be followed by an optimal trajectory toward the origin. If the initial states are on the switching curve, then there will be no switches, and control will be either $+1$ or -1.

   First solve for the system trajectory. Assuming $u = 1$:

   $$\begin{aligned} x_2(t) &= 1 + (x_{20} - 1)e^{-t}, \\ x_1(t) &= e^{-2t}[x_{10} + e^{2t_f} - 1 + 2(e^{tf} - 1)(x_{20} - 1)]. \end{aligned}$$

   To determine the part of the switching curve that corresponds to $u = 1$, we set $x_2(t_f) = x_1(t_f) = 0$ above, and eliminate t_f: $e^{t_f} = 1 - x_{20}$ (clearly $x_{20} < 0$). Substituting this in the second equation above, we obtain $x_{10} = (x_{20})^2$. This is the locus of initial points, with $x_{20} < 0$, from which origin can be reached by $u = 1$. Similarly, for $u = -1$ we obtain $x_{10} = -(x_{20})^2$, with $x_{20} \geq 0$. Hence the switching curve is

   $$x_1 + x_2|x_2| = 0.$$

2. Similar as the previous problem, there is at most one switch. Also for the given linear transformation, $x(t)$ obviously satisfies the system equations given in the previous problem. Substituting $x_1 = y_1$ and $x_2 = \frac{1}{2}(y_2 + 2y_1)$ back to the above solution, the switching curve for $y(t)$ is:

   $$y_1 + \frac{1}{4}|y_2 + 2y_1|(y_2 + 2y_1) = 0.$$

3. The Hamiltonian is $H = p_1(x_2 + u_1) + p_2(-x_1 + u_2)$. This gives the HJ equations

   $$\begin{aligned} \dot{p}_1 &= p_2, \\ \dot{p}_2 &= -p_1. \end{aligned}$$

   Solving them gives $p_1(t) = A\sin(t) + B\cos(t)$ and $p_2 = A\cos(t) - B\sin(t)$. The optimal controls are

   $$u_1^* = -\mathrm{sign}(p_1); \quad u_2^* = \mathrm{sign}(p_2).$$

   The minimum time condition: $H|_{t_f} = -1$ gives $p_1(t_f) + p_2(t_f) = 1$ since $x(t_f) = 0$. Therefore $(A - B)\sin(t_f) + (A + B)\cos(t_f) = 1$. Since p_1 and p_2 are out of phase by $\pi/2$, u_1^* and u_2^* will not switch simultaneously but at alternating times, $\pi/2$ apart. The controls take four possible values: $(1, 1), (1, -1), (-1, 1), (-1, -1)$. For each such choice, the trajectories that make up the phase diagram are circular:

   $$(x_2 + u_1)^2 + (x_1 - u_2)^2 = c^2$$

   where c^2 is a constant determined by the initial conditions. The picture is as Figure 5.3-1 in the handout of Sage and White, but with switching curves not only on the x_1-axis but also on the x_2-axis. There is no easy upper bound on the number of switches.

4. The state may reach $x = 0$ at $t_f < 1$ or $t_f > 0$. Also, because of the hard bound on control, controllability to the origin cannot be achieved in arbitrary small time. However since under $u = \pm 1$, the trajectories on the x_1-x_2 plane:

$$(x_2)^2 + 2x_2 u + 2x_1 u = c$$

cover the entire plane, this means that the origin can be reached from any initial state in some (bounded) time, by a single switch. The switching curve is

$$\begin{aligned} (u = 1): \quad & (x_2)^2 + 2x_2 + 2x_1 = 0, \quad x_1 < 0, x_2 > 0, \\ (u = -1): \quad & (x_2)^2 - 2x_2 - 2x_1 = 0, \quad x_1 > 0, x_2 < 0. \end{aligned}$$

Since there is no constraint on t_f, we first minimize the quadratic cost above with free terminal state at $t = 1$, and then use the control $u = \pm 1$ as described above (if needed). The Hamiltonian is $H = p_1(x_2 + u) - p_2 u + \frac{1}{2}x_1^2$. The co-state equations are

$$\begin{aligned} \dot{p}_1 &= -x_1, \quad p_1(1) = 0, \\ \dot{p}_2 &= -p_1, \quad p_2(1) = 0, \end{aligned}$$

and the optimal control is $u^* = \text{sign}(p_2 - p_1)$, provided that $p_2 \neq p_1$.

If $p_2 = p_1$ on any nonzero interval, $\dot{p}_2 = \dot{p}_1 = 0 \Rightarrow p_1 = x_1 = 0 \Rightarrow x_2 + u = 0$. Hence a candidate singular control is

$$u = -x_2.$$

Note that under this control, $\dot{x}_1 = 0$, and hence it makes sense to switch to this control at time $t_s < 0$ once $x_1(t_s) = 0$. One can verify that this solution satisfies the higher-order necessary condition $-\frac{\partial}{\partial u}\left(\frac{d^2}{dt^2}H_u\right) = 1 > 0$.

Now prior to t_s, we have to use $u = -1$ (if $x_{10} > 0$) or $u = +1$ (if $x_{10} < 0$), which will drive $x_1(t)$ monotonically to $x(t_s) = 0$, assuming that there exists such a $t_s < 1$. From that point, we switch to $u = -x_2$, until $t = 1$. Then we use the switching curve to bring x_2 also to zero, which clearly also minimize the time to reach the origin. If no such $t_s < 1$ exists, then clearly there will be no need for a switch to the singular control until time $t = 1$. Hence we only have to use $u = -1$ or $u = +1$ depending on whether $x_{10} > 0$ or $x_{10} < 0$. This way we would also have minimized the time it takes to reach the origin, t_f, while minimizing the given performance index.

5. The Hamiltonian is $H = x_1^2 + |u| + p_1(-x_1 + x_2) + p_2(-2x_2 + u)$ and the costate equations are

$$\begin{aligned} \dot{p}_1 &= p_1 - 2x_1, \quad p_1(t_f) = 0, \\ \dot{p}_2 &= -p_1 + 2p_2, \quad p_2(t_f) = 0. \end{aligned}$$

Optimal control is

$$u^* = \begin{cases} -1 & p_2 > 1, \\ +1 & p_2 < -1, \\ 0 & -1 < p_2 < 1. \end{cases}$$

For $u = 1$, the state trajectory is $x_2 = A(x_1 + x_2 - 1)^2 + 1/2$ where A is a constant that depends on the initial conditions. Similarly, for $u = -1$, the trajectory is $x_2 = B(x_1 + x_2 + 1)^2 - 1/2$ where B is a constant. Finally, for $u = 0$, we have $x_2 = C(x_1 + x_2)^2$. Notice that around a neighborhood of the terminal time t_f, we have to choose $u = 0$ and follow the third type of trajectory since $p(t_f) = 0$.

The above three types of trajectories completely describe the optimal solution assuming that a singular solution does not exist. We now need to show this is indeed the case. Suppose that $p_2 = 1$ and $\dot{p}_2 = 0$ for some time t. Then, from the second costate equation, $p_1 = 2 \Rightarrow \dot{p}_1 = 0 \Rightarrow x_1 = 1 \Rightarrow x_2 = 1 \Rightarrow u = 2$ in a neighborhood of t where $p_2 = 1$. Since $|u| \leq 1$, this is a contradiction. Likewise, it can be shown that it is not possible to have $p_2 = -1$ and $\dot{p}_2 = 0$.

6. (a) The HJB equation, after minimizing with respect to the control u, is

$$\begin{cases} -\frac{\partial V}{\partial t} = -\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 - \frac{\partial V}{\partial x}(1+x), \\ V(1,x) = x^2. \end{cases}$$

This admits the solution $V(t,x) = p(t)x^2 + 2k(t)x + m(t)$ where

$$\begin{aligned} &\dot{p} - p^2 - 2p = 0, \quad p(1) = 1, \\ &\dot{k} - kp - p - k = 0, \quad k(1) = 0, \\ &\dot{m} - k^2 - 2k = 0, \quad m(1) = 0. \end{aligned}$$

This gives the solution

$$p(t) = \frac{2e^{2t}}{3e^2 - e^{2t}}, \quad k(t) = \frac{2e^t(e - e^t)}{3e^2 - e^{2t}},$$

and the optimal feedback control is

$$u^*(t) = -p(t)x - k(t).$$

(b) Let $x^*(t)$ be the optimal trajectory that solves the

$$\dot{x}^* = -(1 + p(t))x^* - k(t) - 1; \quad x^*(0) = 1.$$

Then we can write

$$u^*(t) = -p(t)x - k(t) = \left[-p(t) - \frac{k(t)}{x^*(t)}\right] x(t)$$

as long as $x^*(t) \neq 0$. However, since such a $x^*(t)$ needs to be computed in advance and depends on the initial conditions, the above control is not state feedback control in its strict sense.

Please report to yima@uiuc.edu if you find any typos.