ECE 553, Spring 2008 **Problem Set #4 Solution**

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Solutions:

1. (a) Since $P(t) = \Lambda(t)X^{-1}(t)$, differentiate it to obtain

$$\begin{aligned}\dot{P}(t) &= \dot{\Lambda}(t)X^{-1} + \Lambda(t)X^{-1}(t)\dot{X}(t)X^{-1}(t) \\ &= -Q - A^T Y(t)X^{-1}(t) - Y(t)X^{-1}(t)A + YX^{-1}(t)BR^{-1}B^T Y(t)X^{-1}(t) \\ &= -Q - A^T P - PA + PBR^{-1}B^T P.\end{aligned}$$

The boundary condition is $P(t_f) = \Lambda(t_f)X^{-1}(t_f) = S$. Since S is symmetric, it is trivial to see that $P^T(t)$ is also a solution to the Riccati equation with the same boundary condition – by simply transposing the Riccati equation. Since a solution to an ODE is unique, we must have $P(t) = P^T(t)$.

(b) From the boundary condition at the terminal time t_f, $\lambda(t_f) = Sx(t_f)$, then

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} x(t_f)$$

must be the only solution to the H-J equation with the given boundary condition. From the definition of $P(t)$, we have $\Lambda(t) = P(t)X(t)$. Therefore, we have

$$\lambda(t) = \Lambda(t)x(t_f) = P(t)X(t)x(t_f) = P(t)x(t).$$

2. The answer is to follow the routine given in the handout. I do not elaborate it here.

3. (a) Construct the extremal piecewise over the sub-intervals $[0, 1/2]$ and $[1/2, 1]$, while making sure that it remains continuous at the point $t = 1/2$. The Euler-Lagrange equation (which applies to both intervals) is

$$\frac{d}{dt}[2(\dot{x} - 1/3)] = 0 \quad \Rightarrow \quad \dot{x} = c_1, \quad \Rightarrow \quad x(t) = c_1 t + c_2.$$

Over the first interval, the given boundary conditions lead to $c_2 = 0, c_1 = 2$, and over the second interval, they lead to $c_1 = 0, c_2 = 1$. Hence the extremal for x is

$$x^o(t) = \begin{cases} 2t, & \text{if} \quad t \in [0, 1/2], \\ 1, & \text{if} \quad t \in [1/2, 1]. \end{cases}$$

Since $\phi_{\dot{x}\dot{x}} = 2 > 0$, the strengthened Legendre condition is satisfied.

(b) Because the mid-point constraint, which was forced on the solution, we cannot expect the solution is overall extremal for the interval $[0, 1]$ and thus the Weierstrass-Erdmann corner conditions are not necessarily satisfied at the broken point $t = 1/2$. In the present case, it is not. You may also check this by solving the minimization problem without the mid-point condition and compare the resulting values of $J(x)$.

4. (a) The optimal control is given by $u^*(t) = -B^T P(t) x(t)$ with

$$B = [0,1]^T, \quad \dot{P} = A^T P + PA - PBB^T P = 0, \quad P(t_f) = \begin{bmatrix} s_{11} & 0 \\ 0 & s_{22} \end{bmatrix}.$$

Let $M = P^{-1}$, which satisfies

$$\dot{M} = MA^T + AM - BB^T = 0, \quad M(t_f) = \begin{bmatrix} 1/s_{11} & 0 \\ 0 & 1/s_{22} \end{bmatrix}.$$

If $M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_4 \end{bmatrix}$, solving the above ODE, we get

$$\begin{aligned} M_4 &= -(t - t_f) + 1/s_{22}, \\ M_2 &= -\frac{1}{2}(t-t_f)^2 + (t-t_f)/s_{22} + \frac{1}{2}t_f^2, \\ M_1 &= -\frac{1}{3}(t-t_f)^3 + (t-t_f)/s_{22} + 1/s_{11}. \end{aligned}$$

Let

$$D = \det(M) = M_1 M_4 - M_2^2 = \left(\frac{1}{3}(t_f - t)^3 + 1/s_{11}\right)(t_f - t + 1/s_{22}) - \frac{1}{4}(t_f - t)^4.$$

Then

$$P = S^{-1} = \frac{1}{D}\begin{bmatrix} M_4 & -M_2 \\ -M_2 & M_1 \end{bmatrix}.$$

Hence

$$u^*(t) = -\left(\frac{-M_2}{D}, \frac{M_1}{D}\right) x(t).$$

(b)

$$\begin{aligned} s_{22} \to 0: \quad & u^*(t) = -\left(\frac{t_f - t}{\frac{1}{3}(t_f-t)^3 + 1/s_{11}}, \frac{(t_f-t)^2}{\frac{1}{3}(t_f-t)^3 + 1/s_{11}}\right) x \\ s_{11} \to 0: \quad & u^*(t) = -\left(0, \frac{1}{t_f - t + 1/s_{22}}\right) x \\ t_f \to \infty: \quad & u^*(t) \equiv 0. \end{aligned}$$

The last case is expected since there is no intermediate cost on state. The optimal cost for $t_f < \infty$ is $J^*_{t_f} = \frac{1}{2}x^T(0)P(0,t_f)x(0)$ where $P(0,t_f)$ is the solution of the Riccati equation, as given above, at $t = 0$, and with t_f considered as a parameter. It is not difficult to see that with fixed s_{11} and s_{22}, $P(0,t_f)$ converges to the zero matrix. so that $\lim_{t_f \to \infty} J^*_{t_f} = 0$. Note, however, that if the limiting control $u^* = 0$ is used, then the corresponding cost would diverge since the open-loop system is unstable (x goes to infinity): $J^* = \infty$.

This is a lesson we learned in ECE415: if the system is not observable (since $q_{11} = q_{22} = 0$), we usually cannot use the limit of optimal controls as the optimal control for the infinite horizon problem.

(c) In this case we can always assume $s_{11} = s_{22}$ since from ECE415 we know that the optimal control is going to be stabilizing, *i.e.* $x \to 0$ as $t \to \infty$. The optimal control $u^* = -B^T \bar{P} x$ is given by the algebraic Riccati equation:

$$\bar{P}A + A^T\bar{P} - \bar{P}BB^T\bar{P} + Q = 0.$$

Solve this equation to get $P_1 = P_3 = \sqrt{3}, P_2 = 1$. Therefore, $u^* = -x_1 - \sqrt{3}x_2$. The closed-loop system under this control is asymptotically stable (as expected).

5. The E-L is

$$2\frac{d}{dt}\dot{x} = -2x \quad \Rightarrow \ddot{x} = -x.$$

With the given boundary condition, it admits the unique solution $x(t) = -\sin(t)$.

(a) Jacobi's equation is $\ddot{\eta} + \eta = 0$. Taking $\eta(0) = 0$, it admits the type of solutions: $\eta(t) = A\sin(t)$. Clearly the point $t = \pi$ is conjugate to $t = 0$, since $A\sin(\pi) = 0$ for all A. Hence, the given extremal cannot be an optimal solution to the problem. Since it was the only extremal, it follows that there is no solution to this problem.

(b) It is straightforward to show that the second order variation is

$$\delta^2 J = 2\int_0^{3\pi/2} \left(\dot{\eta}(t)^2 - \eta(t)^2\right)\, dt.$$

With the given $\eta(t) = \sin(2t/3)$, we get $\delta^2 J = -5\pi/6 < 0$. Hence $J(x^o + \epsilon\eta) - J(x^o) \approx \frac{\epsilon^2}{2}\delta^2 J(x^9; \eta) = -\frac{5\pi}{12}\epsilon^2$ for small enough ϵ. Therefore, x^o cannot be a minimum.

Please report to yima@uiuc.edu if you find any typos.