ECE 553, Spring 2008 **Problem Set #3 Solution**

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Solutions:

1. The problem is quite simple once you realize that the terminal time condition $t = p(x,t)$ is $x(t_f) = c(t_f)$ and then $t = p(c(t),t)$. Therefore, $1 = \frac{\partial p}{\partial c}\dot{c} + \frac{\partial p}{\partial t}$ yields

$$\dot{c} = \left(1 - \frac{\partial p}{\partial t}\right) / \frac{\partial p}{\partial c}.$$

Then you can use $\dot{c}$ to substitute in the boundary condition for the free terminal time case

$$\phi_{\dot{x}}(\dot{c} - \dot{x}) + \phi\big|_{t=t_f} = 0.$$

2. (a) Following the derivation for the isoperimetric problem we did in class, in the multiple integral constraints case, we get

$$\int_{t_0}^{t_f} \left(\phi_x - \frac{d}{dt}\phi_{\dot{x}}\right) \eta \; dt = 0, \quad \forall \eta$$
$$s.t. \quad \int_{t_0}^{t_f} \left((\psi_i)_x - \frac{d}{dt}(\psi_i)_{\dot{x}}\right) \eta \; dt = 0, \quad i = 1, \ldots, m.$$

Since the function $\left(\phi_x - \frac{d}{dt}\phi_{\dot{x}}\right)$ is orthogonal to all functions (η) in the kernel of the linear operators:

$$L_i : \; \eta \mapsto \int_{t_0}^{t_f} \left((\psi_i)_x - \frac{d}{dt}(\psi_i)_{\dot{x}}\right) \eta \; dt,$$

it must be in the range of these linear operators (according to the finite rank fundamental lemma). Hence $\left(\phi_x - \frac{d}{dt}\phi_{\dot{x}}\right)$ must be a linear combination of $\left((\psi_i)_x - \frac{d}{dt}(\psi_i)_{\dot{x}}\right), i = 1, 2, \ldots, m$. That is, there exists $\lambda_i \in \mathbb{R}$ such that

$$\left(\phi_x - \frac{d}{dt}\phi_{\dot{x}}\right) = -\sum_{i=1}^{m} \lambda_i \left((\psi_i)_x - \frac{d}{dt}(\psi_i)_{\dot{x}}\right).$$

(b) If the end point is free, a similar situation occurs between $\phi_{\dot{x}}|_{t_f}$ and $(\psi_i)_{\dot{x}}|_{t_f}$ where the finite rank fundamental lemma can be applied again.

(c) Simply define

$$L = \phi + \sum_{i=1}^{m} \lambda_i \psi_i.$$

The E-L equation associated with L directly leads to the same equations as above.

3. First eliminate the dependence of J on x and $\dot{x}$, we obtain

$$\min \tilde{J}(y) = \int_0^1 [\ddot{y}^2 + \dot{y}^2 + 2\dot{y}y] \; dt.$$

The E-L equation for this problem is

$$\frac{d^2}{dt^2}(2\ddot{y}) - \frac{d}{dt}(2\dot{y} + 2y) + 2\dot{y} = 0, \quad \dot{y}(0) = \dot{y}(1) = y(1) = 0, y(0) = 1.$$

Integrating the above equation once and substituting $x = \dot{y}$, we get an equivalent equation

$$\ddot{x} = x + c_1.$$

Solve the above equations with given boundary conditions we have

$$\begin{aligned} x^o(t) &= A\cosh(t) + B\sinh(t) - A, \\ y^o(t) &= A\sinh(t) + B\cosh(t) - At - B + 1, \end{aligned}$$

where

$$A = \frac{\sinh(1)}{2(1-\cosh(1)) + \sinh(1)}, \quad B = \left(\frac{1-\cosh(1)}{\sinh(1)}\right) A.$$

If J does not have the cross term $2xy$, then the associated E-L equation reduces to the same differential equation ($\ddot{x} = x + c$), leading to the same solution as above. Hence, two different cost functions could result in the same extremal.

4. I gave three solutions to this problem in class.

5. Again, one may have multiple ways of solving this problem. The following is a possible one.

$$\min J = \int_0^{t_f} [\dot{x}_2]^2 \, dt \quad s.t. \quad \dot{x}_1 = x_2, \quad x_1(0) = x_2(0) = 1, x_1(t_f) = -t_f^2, t_f \ free.$$

E-L equations are

$$2\ddot{x}_2 = -\lambda, \quad \dot{\lambda} = 0.$$

This yields

$$\begin{aligned} x_2(t) &= -\frac{1}{4}c_1 t^2 + c_2 t + 1, \\ x_1(t) &= -\frac{1}{12}c_1 t^3 + \frac{1}{2}c_2 t^2 + t + 1. \end{aligned}$$

Transversality:

$$\lambda(t_f)[-2t_f - x_2(t_f)] + \dot{x}_2(t_f)^2 = 0.$$

Boundary condition for the free x_2:

$$2\dot{x}_2(t_f) = 0.$$

In addition to the condition $x_1(t_f) = -t_f^2$. Solving the above three equations we obtain

$$t_f^* = \frac{1+\sqrt{13}}{2}, \quad c_1 = -\frac{30 + 6\sqrt{13}}{5 + 2\sqrt{13}}, \quad c_2 = -\frac{27 + 9\sqrt{13}}{5 + 2\sqrt{13}}$$

and $u^*(t) = c_2 - \frac{1}{2}c_1 t$.

6. Eliminate $\dot{x}$ from the first equation, to arrive at an equivalent cost function:

$$\int_0^1 [\dot{z} - \dot{y}]^2 \; dt$$

to be minimized. This leaves us with only one differential equality constraint, in view of which the Lagrangian is

$$L(\dot{y}, \dot{z}, \dot{w}, \lambda) = [\dot{z} - \dot{y}]^2 + \lambda[\dot{w} + 2\dot{z}].$$

The E-L equations are

$$\frac{d}{dt} L_{\dot{y}} = 0, \quad \frac{d}{dt} L_{\dot{z}} = 0, \quad \frac{d}{dt} L_{\dot{w}} = 0.$$

Solving these subject to the given initial and terminal conditions, and in view of the given two constraints, leads to the family of solutions:

$$x^o(t) = -2t + c, \quad w^o(t) = -2z^o(t), \quad y^o(t) = 2t + z^o(t),$$

where c is an arbitrary constant, and $z^o(t)$ is any continuously differentiable function, satisfying the given boundary conditions.

7. Clearly $J(x) \geq 0$, and it is zero if either $\dot{x}(t) = 0$ or $\dot{x} = 1$ or both. Integrating these, we have

$$x_1(t) = c_1, \quad x_2(t) = t + c_2.$$

Two possible solutions that satisfy the given boundary conditions are

$$x^o(t) = \begin{cases} t, & 0 \leq t \leq 2/3, \\ 2/3, & 2/3 < t \leq 1 \end{cases} \qquad x^*(t) = \begin{cases} 0, & 0 \leq t \leq 1/3, \\ t - 1/3, & 1/3 < t \leq 1 \end{cases}$$

each with a single corner, at $t = 2/3$ and $t = 1/3$, respectively. The W-E corner conditions are:

(a) $\phi_{\dot{x}} = 2\dot{x}[1-\dot{x}]^2 - 2\dot{x}^2[1-\dot{x}]|_{x^o} = 0$ also $|_{x^*} = 0$ first condition holds trivially.

(b) $\phi - \dot{x}\phi_{\dot{x}} = \dot{x}^2[1-\dot{x}][1 - \dot{x} - 2 + \dot{x}] = 0$ which again holds trivially for both $x = x^o$ and $x = x^*$.

Please report to yima@uiuc.edu if you find any typos.