ECE 553, Spring 2008 **Problem Set #2 Solution**

Posted: February 28th, 2008

Solutions:

1. The objective functional is $J(x) = \int_0^1 \ddot{x}^2(t)\, dt$. From the previous homework, the associated E-L equation $\frac{d^2}{dt^2}\ddot{x} = 0$ admits a solution of the general form:

$$x^o(t) = a + bt + ct^2 + et^3.$$

From the given boundary conditions, the coefficients a and b can easily be determined:

$$a = -2\sqrt{2}, \quad b = 5\sqrt{2}.$$

To determine the remaining two coefficients c and e, we need two independent equations at $t = 1$. One is the given end-point condition $x^2(1) + \dot{x}^2(1) = 1$. The other will be obtained by the boundary condition from the variation:

$$\left[(\phi_{\dot{x}} - \frac{d}{dt}\phi_{\ddot{x}})\eta + \phi_{\ddot{x}}\dot{\eta}\right]_{t=1} = 0. \tag{1}$$

However, be aware that here η and $\dot{\eta}$ are not totally unrelated. The admissible variation η satisfies extra constraints at $t = 1$:

$$\frac{d}{d\epsilon}\left[(x^o(1) + \epsilon\eta(1))^2 + (\dot{x}^o(1) + \epsilon\dot{\eta}(1))^2\right]_{\epsilon=0} = 0.$$

It yields

$$x^o(1)\eta(1) + \dot{x}^o(1)\dot{\eta}(1) = 0 \quad \Rightarrow \quad \dot{\eta}(1) = -\frac{x^o(1)}{\dot{x}^o(1)}\eta(1).$$

Substituting this back to equation (1) and in general $\eta(1) \neq 0$, we obtain the equation

$$\dot{x}^o(1)x^{o(3)}(1) + \ddot{x}^o(1)x^o(1) = 0.$$

Using this equation and $x^2(1) + \dot{x}^2(1) = 1$ to solve c and e, we have

$$c = -3\sqrt{2}, \quad e = \sqrt{2}/2.$$

Now differentiate $x^o(t) = t^3\sqrt{2}/2 - t^2 3\sqrt{2} + t5\sqrt{2} - 2\sqrt{2}$ twice to obtain the optimal control

$$u^*(t) = t3\sqrt{2} - 6\sqrt{2}.$$

2. The E-L equation is the same as the above problem, with the given boundary condition $x(0) = \dot{x}(0) = 1$, we have $a = b = 1$ and therefore the solution to $x^o(t)$ is of the form

$$x^o(t) = 1 + t + ct^2 + et^3.$$

To solve the other two coefficients, one can use the transversality conditions for the free terminal time case – and you are encourage to do so by yourself. Since we here have the closed form solution to $x^o(t)$, we can take a more direct approach. With $x^o(t)$ substituting back into $J(x)$, we obtain

$$J(x^o) = \int_0^{t_f} (2c + 6et)^2 \, dt = 4[c^2 t_f + 3ect_f^2 + 3e^2 t_f^3].$$

Then the original problem becomes minimizing the above function (in three unknowns c, e, t_f) subject to the terminal set constraint $x(t_f) = -t_f^2$:

$$1 + t_f + (c+1)t_f^2 + et_f^3 = 0.$$

This minimization problem can be solved by first solving for c in terms of t_f from the above equation, substituting this into $J(x^o)$, and then differentiating the resulting function of e and t_f with respect to these two parameters, and setting the derivatives equal to zero. Some manipulations lead to the unique solutions:

$$t_f^* = \frac{1+\sqrt{13}}{2} = 2.3028, \quad c = -\frac{11+\sqrt{13}}{6} = -2.4343, \quad e = \frac{1+5\sqrt{13}}{54} = 0.3524.$$

The optimal control sought is

$$u^*(t) = \ddot{x}^o(t) = -4.8685 + 2.1142t.$$

3. Let η be an admissible variation with its strong norm $\|\eta\|_s = 1$. Then, $x = x^o + \epsilon\eta$ lead to

$$J(x^o + \epsilon\eta) - J(x^o) = \epsilon \int_{t_0}^{t_f} \left(\phi_x - \frac{d}{dt}\phi_{\dot{x}}\right) \eta(t) \, dt + \epsilon \phi_{\dot{x}} \eta \Big|_{t_0}^{t_f} + \epsilon \psi_x(x(t_f))\eta(t_f) + o(\epsilon).$$

The first term leads to our standard Euler-Lagrange equation

$$\phi_x - \frac{d}{dt}\phi_{\dot{x}} = 0.$$

The rest terms lead to the equation

$$[\phi_{\dot{x}}(t_f) + \psi_x(x(t_f))]\,\eta(t_f) - \phi_{\dot{x}}(t_0)\eta(t_0) = 0$$

for all admissible η. However, be aware that here η is restricted by the given boundary condition $x(t_0) + 2x(t_f) = 1$:

$$\eta(t_0) + 2\eta(t_f) = 0 \quad \Rightarrow \quad \eta(t_0) = -2\eta(t_f).$$

Since $\eta(t_f)$ can be non-zero, we get

$$\phi_{\dot{x}}(t_f) + \psi_x(x(t_f)) + 2\phi_{\dot{x}}(t_0) = 0.$$

This equation, with the given equation $x(t_0) + 2x(t_f) = 1$, specifies the two boundary conditions for the E-L equation.

4. The Euler-Lagrange equation is

$$\frac{d^2}{dt^2}\ddot{x} = x$$

which admits the general solution

$$x(t) = A\cosh(t) + B\sinh(t) + C\sin(t) + D\cos(t).$$

Using the given four boundary conditions, unique values can be obtained for A, B, C, and D:

$$A = \frac{\left(1 - \sinh\frac{\pi}{2}\right)\sinh\frac{\pi}{2}}{\cosh^2\frac{\pi}{2}}, \quad B = \tanh\frac{\pi}{2}, \quad C = -B, \quad D = 1 - A.$$

Please report to yima@uiuc.edu if you find any typos.