Solutions:

1. The objective functional is \( J(x) = \int_0^1 \dot{x}^2(t) \, dt \). From the previous homework, the associated E-L equation \( \frac{d^2}{dt^2} \ddot{x} = 0 \) admits a solution of the general form:

\[
x^o(t) = a + bt + ct^2 + et^3.
\]

From the given boundary conditions, the coefficients \( a \) and \( b \) can easily be determined:

\[
a = -2\sqrt{2}, \quad b = 5\sqrt{2}.
\]

To determine the remaining two coefficients \( c \) and \( e \), we need two independent equations at \( t = 1 \). One is the given end-point condition \( x^2(1) + \dot{x}^2(1) = 1 \). The other will be obtained by the boundary condition from the variation:

\[
\left[ (\phi \ddot{x} - \frac{d}{dt} \phi \dot{x}) \eta + \phi \dot{x} \dot{\eta} \right]_{t=1} = 0.
\] (1)

However, be aware that here \( \eta \) and \( \dot{\eta} \) are not totally unrelated. The admissible variation \( \eta \) satisfies extra constraints at \( t = 1 \):

\[
\frac{d}{d\epsilon} \left[ (x^o(1) + \epsilon \dot{x}(1))^2 + (\dot{x}^o(1) + \epsilon \ddot{x}(1))^2 \right]_{\epsilon=0} = 0.
\]

It yields

\[
x^o(1)\dot{x}(1) + \dot{x}^o(1)\ddot{x}(1) = 0 \quad \Rightarrow \quad \ddot{x}(1) = -\frac{x^o(1)}{\dot{x}^o(1)} \eta(1).
\]

Substituting this back to equation (1) and in general \( \eta(1) \neq 0 \), we obtain the equation

\[
\dot{x}^o(1)x^{o(3)}(1) + \ddot{x}^o(1)x^o(1) = 0.
\]

Using this equation and \( x^2(1) + \dot{x}^2(1) = 1 \) to solve \( c \) and \( e \), we have

\[
c = -3\sqrt{2}, \quad e = \sqrt{2}/2.
\]

Now differentiate \( x^o(t) = t^3\sqrt{2}/2 - t^23\sqrt{2} + t5\sqrt{2} - 2\sqrt{2} \) twice to obtain the optimal control

\[
u^*(t) = t3\sqrt{2} - 6\sqrt{2}.
\]

2. The E-L equation is the same as the above problem, with the given boundary condition \( x(0) = \dot{x}(0) = 1 \), we have \( a = b = 1 \) and therefore the solution to \( x^o(t) \) is of the form

\[
x^o(t) = 1 + t + ct^2 + et^3.
\]

To solve the other two coefficients, one can use the transversality conditions for the free terminal time case – and you are encourage to do so by yourself. Since we here have the
closed form solution to $x^0(t)$, we can take a more direct approach. With $x^0(t)$ substituting back into $J(x)$, we obtain

$$J(x^0) = \int_0^{t_f} (2c + 6et)^2 \, dt = 4[e^2 t_f^3 + 3ect_f^2 + 3e^2 t_f^3].$$

Then the original problem becomes minimizing the above function (in three unknowns $c, e, t_f$) subject to the terminal set constraint $x(t_f) = -t_f^2$:

$$1 + t_f + (c + 1)t_f^2 + et_f^3 = 0.$$

This minimization problem can be solved by first solving for $c$ in terms of $t_f$ from the above equation, substituting this into $J(x^0)$, and then differentiating the resulting function of $e$ and $t_f$ with respect to these two parameters, and setting the derivatives equal to zero. Some manipulations lead to the unique solutions:

$$t_f^* = \frac{1 + \sqrt{13}}{2} = 2.3028, \quad c = -\frac{11 + \sqrt{13}}{6} = -2.4343, \quad e = \frac{1 + 5\sqrt{13}}{54} = 0.3524.$$

The optimal control sought is

$$u^*(t) = \ddot{x^0}(t) = -4.8685 + 2.1142t.$$

3. Let $\eta$ be an admissible variation with its strong norm $||\eta||_s = 1$. Then, $x = x^0 + \epsilon \eta$ lead to

$$J(x^0 + \epsilon \eta) - J(x^0) = \epsilon \int_{t_0}^{t_f} \left( \phi_x - \frac{d}{dt} \phi_x \right) \eta(t) \, dt + \epsilon \phi_x(t_0) + \epsilon \psi_x(x(t_f)) \eta(t_f) + o(\epsilon).$$

The first term leads to our standard Euler-Lagrange equation

$$\phi_x - \frac{d}{dt} \phi_x = 0.$$

The rest terms lead to the equation

$$[\phi_x(t_f) + \psi_x(x(t_f))] \eta(t_f) - \phi_x(t_0) \eta(t_0) = 0$$

for all admissible $\eta$. However, be aware that here $\eta$ is restricted by the given boundary condition $x(t_0) + 2x(t_f) = 1$:

$$\eta(t_0) + 2\eta(t_f) = 0 \quad \Rightarrow \quad \eta(t_0) = -2\eta(t_f).$$

Since $\eta(t_f)$ can be non-zero, we get

$$\phi_x(t_f) + \psi_x(x(t_f)) + 2\phi_x(t_0) = 0.$$

This equation, with the given equation $x(t_0) + 2x(t_f) = 1$, specifies the two boundary conditions for the E-L equation.

4. The Euler-Lagrange equation is

$$\frac{d^2}{dt^2} x(t) = x$$

which admits the general solution

$$x(t) = A \cosh(t) + B \sinh(t) + C \sin(t) + D \cos(t).$$

Using the given four boundary conditions, unique values can be obtained for $A, B, C, D$:

$$A = \left(1 - \sinh \frac{\pi}{2}\right) \sinh \frac{\pi}{2} \cosh \frac{\pi}{2}, \quad B = \tanh \frac{\pi}{2}, \quad C = -B, \quad D = 1 - A.$$

Please report to yima@uiuc.edu if you find any typos.