ECE 553, Spring 2008 **Problem Set #1 Solution**

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Solutions:

1. (a) $\|\lambda t(1-t)\|_{C^1} = |\lambda| \le 1/2$.
   (b) $\|\lambda t(1-t)\|_{\infty} = \frac{1}{4}|\lambda| \le 1/2 \Rightarrow |\lambda| \le 2$.
   (c) The weak neighborhood is larger than the strong neighborhood.

2. (a) We derive the first order condition for the special case $n = 2$. The general case is similar but more tedious. Around any extremal y^o, we consider a small variation to it as $y(t) = y^o(t) + \epsilon\eta(t)$ where η is at least twice continuously differentiable. Then for $J(y) = J(y^o + \epsilon\eta)$ as a function of ϵ, we have

$$\delta J(x^o;\eta) = \frac{\partial J}{\partial \epsilon} = \int_a^b [L_x\eta + L_{\dot{x}}\dot{\eta} + L_{\ddot{x}}\ddot{\eta}]\ dt = 0, \quad \forall \eta \in N$$

where $L_x = \frac{\partial L}{\partial x}$ and similarly for $L_{\dot{x}}$ and $L_{\ddot{x}}$, and N is the set of admissible variations. Integrating by part, we get

$$\int_a^b [L_x\eta + L_{\dot{x}}\dot{\eta} + L_{\ddot{x}}\ddot{\eta}]\ dt = \int_a^b [L_x\eta + (L_{\dot{x}} - \frac{d}{dt}L_{\ddot{x}})\dot{\eta}]\ dt + \dot{\eta}L_{\ddot{x}}\Big|_a^b$$

$$= \int_a^b [L_x - \frac{d}{dt}L_{\dot{x}} + \frac{d^2}{dt^2}L_{\ddot{x}}]\eta\ dt + (L_{\dot{x}} - \frac{d}{dt}L_{\ddot{x}})\eta\Big|_a^b + L_{\ddot{x}}\dot{\eta}\Big|_a^b$$

Now to simplify the boundary conditions, let us assume the end points are fixed, *i.e.* $\eta(a) = \eta(b) = 0$. Furthermore, to eliminate the last term in the previous expression, we make the assumption that the admissible variations are such that $\dot{\eta}(a) = \dot{\eta}(b) = 0$ too. Therefore, if the function L is thrice continuously differentiable in its arguments, the first order condition leads to the following differential equation:

$$\frac{d^2}{dt^2}L_{\ddot{x}} - \frac{d}{dt}L_{\dot{x}} + L_x = 0, \quad \text{at } x = x^o.$$

Notice that if x and $\dot{x}$ are not fixed at the end points, we further have the boundary conditions:

$$L_{\ddot{x}}\big|_{a,b} = 0, \quad L_{\dot{x}} - \frac{d}{dt}L_{\ddot{x}}\big|_{a,b} = 0. \qquad (1)$$

You should convince yourself why this is the case.

   (b) Let $\mathbf{z} = [x,y]^T \in \mathbb{R}^2$. Then the original problem becomes $\min J(\mathbf{z}) = \int_a^b L(\mathbf{z}(t), \dot{\mathbf{z}}(t))\ dt$, $\mathbf{z}(\cdot) \in C^1[a,b] \times C^1[a,b]$. Applying the same variational calculus to $\mathbf{z}(\cdot)$ as we did in class to $\min J(x) = \int_a^b \phi(x(t), \dot{x}(t))\ dt, x(\cdot) \in C^1[a,b]$, we should get the E-L equation:

$$\frac{\partial L}{\partial \mathbf{z}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{z}}} = 0$$

where $\frac{\partial L}{\partial \mathbf{z}} \doteq [\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}]^T$ and similarly for $\frac{\partial L}{\partial \dot{\mathbf{z}}}$.

3. (a) Let $\phi(y, \dot{y}, x) = y\sqrt{1 + \dot{y}}$. Since ϕ does not depend on x, the E-L equation in this special case is equivalent to

$$\phi - \dot{y}\phi_{\dot{y}} = c$$

for some constant c.

(b) The above differential equation gives

$$y = c\sqrt{1+\dot{y}^2} \quad \Rightarrow \quad \dot{y} = \sqrt{\frac{y^2 - c^2}{c^2}} \quad \Rightarrow \quad y = c \cdot \cosh\left(\frac{x-a}{c}\right).$$

Depending on the boundary conditions $y(a), y(b)$, the might not exist a solution.

4. (a) The E-L equation associated to this problem is

$$\frac{d}{dt}(\dot{x}x^2) = -x(1 - \dot{x}^2), \quad x(0) = x(\pi) = 0.$$

(b) Clearly $x^o = 0$ is a solution to this equation and $J(x^o) = 0$.

(c) First, notice that $x_{(n)}$ is admissible since $x_{(n)}(0) = x_{(n)}(\pi) = 0$. Since $\|x_{(n)}\|_\infty = 1/\sqrt{n}$, when $n \geq \frac{1}{\epsilon_0^2}$, $x_{(n)} \in N^w_{\epsilon_0}(x^o)$. Second,

$$J(x_{(n)}) = \int_0^\pi x^2_{(n)}(t)[1 - \dot{x}^2_{(n)}(t)]\ dt = \frac{1}{n}\int_0^\pi \sin^2(nt) - \int_0^\pi \sin^2(nt)\cos^2(nt)\ dt.$$

The second term above is uniformly (for all $n > 0$) bounded away from zero, but the first term can be made arbitrarily close to zero by choosing n sufficiently large. Hence there exists an n_o such that for all $n > n_o$ we have $J(x_{(n)} < 0$. This complete the proof that $x^o = 0$ cannot be a local (strong) minimum since there are $x_{(n)}$'s arbitrarily close (in the weak norm sense) but with strictly smaller J value.

5. From the given conditions, we have

$$\int_a^b f(t)\dot{\eta}(t)\ dt = \int_a^b f(t)(c - f(t))\ dt = 0.$$

Adding to this the zero quantity $-c\int_a^b (c - f(t))\ dt$, we obtain

$$\int_a^b f(t)\dot{\eta}(t)\ dt = -\int_a^b (c - f(t))^2\ dt = 0.$$

Therefore, $f(t) \equiv c$ almost everywhere for $t \in [a, b]$.

Please report to yima@uiuc.edu if you find any typos.