Homework is due at the beginning of class on the due day!

Problems:

1. In problem 6 of homework 5, replace the deterministic state equation by the stochastic one:

   \[ \dot{x} = -x + u - 1 + w, \quad x(0) = 1 \]

   where \( w \) is the standard white Gaussian noise (WGN), and also replace \( J \) by its expected value. Derive the state-feedback controller that minimizes \( E\{J\} \), and obtain an expression for the minimum value of \( E\{J\} \).

2. Consider a variation of the standard LQG control problem with perfect state measurements discussed in class, where now the state \( x(t), t \geq t_0 \) is not available continuously, but only sample-wise. That is, we have a partition of the time interval \([t_0, t_f] = [t_0, t_1] \cup [t_1, t_2] \ldots \cup [t_{n-1}, t_n = t_f]\), and only sampled values \( x(t_i), t_i = 0, 1, \ldots, n - 1, \) of the state are available. Permissible controls are now assumed to be in the form \( u(t, x(t_i)) \) for \( t \in [t_i, t_{i+1}) \). Obtain the optimal control policy. [Hint: try to decouple the “deterministic” part and the “stochastic” part of the process during each interval.]

3. Let \( a \) be a scalar Gaussian random variable with mean zero an variance \( \rho \), and \( x \) be a signal described by

   \[ \dot{x} = a, \quad x(0) = 1, \dot{x}(0) = 1. \]

   The signal \( x \) is not measured directly, but a noisy version \( y \) of \( x \) is available:

   \[ y = x + v, \]

   where \( v \) is the standard white Gaussian noise. Construct a Kalman filter for estimating \( x(t) \), and study the limiting behavior of the Kalman gain and the error (co)variance as \( t \to \infty \).

4. Consider the scalar LQG optimal control problem with dynamics

   \[ \dot{x} = \frac{t^2}{2} + u + w, \quad x(0) = 1 \]

   and cost function

   \[ J = E\left\{ x(1)^2 + \int_0^1 u(t)^2 \, dt \right\}, \]

   where \( w \) is the standard white Gaussian noise.

   (a) Determine the optimal control policy under perfect state information.

   (b) Do the same under open-loop information (no information about the actual state available).

   (c) Let \( J_f \) denote the minimum value of \( J \) in (a) and \( J_o \) denote the minimum value in (b). Evaluate \( J_o - J_f \) without computing \( J_o \) and \( J_f \) separately.
5. You are given a scalar stochastic system:
\[ \dot{x} = u + 1.5w, \quad x(0) = 1, \]
and noisy state measurements:
\[ y = x + v, \quad t \geq 0, \]
where \( w \) and \( v \) are independent standard white Gaussian noises. Let the cost function be exponentially discounted:
\[ J = E \left\{ \int_0^\infty e^{-2t} \left[ x^2 + u^2 \right] dt \right\}. \]
Find a stationary controller \( u^*(\tilde{x}(t)) \), that minimizes \( J \). What is the corresponding value of \( J \)?