ECE 553, Spring 2008 — Problem Set #6

Issued: April 15th, 2008 — **Due: April 23rd, 2008**

Primary Reading: Lecture notes on dynamical programming and LQR control.

Homework is due at the beginning of class on the due day!

Problems:

1. **Singular arc** (10 points)
   Given the following nonlinear system:

   $$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - x_1 u,$$

   we wish to pick a control that drives the system state from a given point ($x_1(0) = x_{10}, x_2(0) = x_{20}$) to the terminal manifold $x_1(t_f)^2 + x_2(t_f)^2 = 1$ (unit circle), while minimizing the quadratic cost:

   $$J(u; t_f) = \int_0^{t_f} [\alpha^2 x_1(t)^2 + x_2(t)^2] \, dt$$

   where t_f is free and α is a given nonnegative constant. Show that this optimal control problem features a singular solution, where the optimal singular trajectory is

   $$\alpha x_1(t) = x_2(t), \quad \alpha x_1(t) = -x_2(t),$$

   and the optimal control on the singular trajectory is

   $$u(t) = -\frac{\alpha^2 x_1(t) + x_2(t)}{x_1(t)}.$$

2. **A nonlinear system** (10 points)
   Given the optimal control problem for a scalar nonlinear system:

   $$\dot{x} = xu, \quad x(0) = 1,$$
   $$J(u) = x(1)^2 + \int_0^1 [x(t)u(t)]^2 \, dt,$$

   obtain the optimal feedback solution by solving the associated HJB equation. [Hint: First show that the HJB partial differential equation admits a solution that is quadratic in x.]

3. Consider the following two-dimensional standard (infinite-horizon) regulator problem with perfect state measurements:

   $$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & \alpha \end{bmatrix} x + \begin{bmatrix} 1 \\ \beta \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
   $$J(u) = \int_0^{\infty} \left[(\rho x_1(t) + x_2(t))^2 + u(t)^2 \right] \, dt$$

   where α, β, ρ are three scalar parameters. Find the complete range of values for these parameters such that each statement below will be valid:

(a) The control problem admits a unique optimum stabilizing state-feedback controller;

(b) The associated algebraic Riccati equation (ARE) admits a unique positive definite solution;

(c) There exists a stabilizing state-feedback controller $u(x) = -[1, \beta]\bar{P}x$, where $\bar{P}$ solves the ARE associated with this problem.

4. Same as the above problem, with only the cost function replaced by

$$J(u) = \int_0^\infty \left[(\rho x_1(t) + x_2(t))^2 + x_2(t)^2 + 2x_1(t)u(t) + u(t)^2 \right] \, dt.$$

[Hint: consider a change of variable back to the standard form.]