Homework is due at the beginning of class on the due day!

Problems:

1. **Singular arc** (10 points)
   Given the following nonlinear system:
   \[
   \begin{align*}
   \dot{x}_1 &= x_2, \\
   \dot{x}_2 &= -x_2 - x_1 u,
   \end{align*}
   \]
   we wish to pick a control that drives the system state from a given point \((x_1(0) = x_{10}, x_2(0) = x_{20})\) to the terminal manifold \(x_1(t_f)^2 + x_2(t_f)^2 = 1\) (unit circle), while minimizing the quadratic cost:
   \[
   J(u; t_f) = \int_0^{t_f} \left[ \alpha^2 x_1(t)^2 + x_2(t)^2 \right] dt
   \]
   where \(t_f\) is free and \(\alpha\) is a given nonnegative constant. Show that this optimal control problem features a singular solution, where the optimal singular trajectory is
   \[
   \alpha x_1(t) = x_2(t), \quad \alpha x_1(t) = -x_2(t),
   \]
   and the optimal control on the singular trajectory is
   \[
   u(t) = -\frac{\alpha^2 x_1(t) + x_2(t)}{x_1(t)}.
   \]

2. **A nonlinear system** (10 points)
   Given the optimal control problem for a scalar nonlinear system:
   \[
   \begin{align*}
   \dot{x} &= xu, \\
   x(0) &= 1,
   \end{align*}
   \]
   \[
   J(u) = x(1)^2 + \int_0^1 [x(t)u(t)]^2 dt,
   \]
   obtain the optimal feedback solution by solving the associated HJB equation. [Hint: First show that the HJB partial differential equation admits a solution that is quadratic in \(x\).]

3. Consider the following two-dimensional standard (infinite-horizon) regulator problem with perfect state measurements:
   \[
   \begin{align*}
   \dot{x} &= \begin{bmatrix} -1 & 1 \\ 0 & \alpha \end{bmatrix} x + \begin{bmatrix} 1 \\ \beta \end{bmatrix} u, \\
   x(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix},
   \end{align*}
   \]
   \[
   J(u) = \int_0^\infty \left[ (\rho x_1(t) + x_2(t))^2 + u(t)^2 \right] dt
   \]
   where \(\alpha, \beta, \rho\) are three scalar parameters. Find the complete range of values for these parameters such that each statement below will be valid:
(a) The control problem admits a unique optimum stabilizing state-feedback controller;
(b) The associated algebraic Riccati equation (ARE) admits a unique positive definite solution;
(c) There exists a stabilizing state-feedback controller $u(x) = -[1, \beta]\bar{P}x$, where $\bar{P}$ solves the ARE associated with this problem.

4. Same as the above problem, with only the cost function replaced by

$$ J(u) = \int_0^\infty \left[ (\rho x_1(t) + x_2(t))^2 + x_2(t)^2 + 2x_1(t)u(t) + u(t)^2 \right] \, dt. $$

[Hint: consider a change of variable back to the standard form.]