ECE 553, Spring 2008 — Problem Set #5

Issued: April 1st, 2008 — **Due: April 9th, 2008**

Primary Reading: Lecture notes on maximum principle and dynamical programming.

Homework is due at the beginning of class on the due day!

Problems:

1. **Bang-bang control 1** (10 points)
   For the second-order system:
   $$\dot{x} = \begin{bmatrix} -2 & 2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = x_0,$$
   we wish to drive the system state $x = [x_1, x_2]^T$ from a given initial state x_0 to the origin in minimum time, by using a control u that is bounded in magnitude by 1; *i.e.*$|u(t)| \leq 1$ for all $t \geq 0$. Show that the optimal control is of bang-bang type, and switches sign at most once. Obtain an expression for the switching curve.

2. **Bang-bang control 2** (10 points)
   Consider the same question as in Problem 22 above, this time for the system:
   $$\dot{y} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} y + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \quad y(0) = y_0.$$
   In addition, show that the two systems are related by the linear transformation:
   $$y(t) = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix} x(t).$$
   How does this linear transformation affect the switching curves of the two systems?

3. **Bang-bang control 3** (10 points)
   Consider the system whose state equations are
   $$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = -x_1 + u_2,$$
   with $|u_1| \leq 1$ and $|u_2| \leq 1$. Obtain the controller that would transfer an arbitrary initial state to the origin in minimum time. Sketch the general nature of the switching curves.

4. (10 points)
   Consider the system whose state equations are
   $$\dot{x}_1 = x_2 + u, \quad \dot{x}_2 = -u,$$
   with $|u| \leq 1$, which we wish to transfer from an arbitrary initial state to the origin while minimizing the value of the performance index
   $$J = \int_0^1 x_1^2(t)dt.$$
   Discuss the feasibility of such an objective, and if feasible obtain the optimal control that will do the transfer.

5. (10 points)
   Discuss the nature of the solution to the following optimal control problem:

$$\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -2x_2 + u, \quad |u| \le 1,$$

$$\min J(u; t_f) = \int_0^{t_f} \left[x_1^2(t) + |u(t)|\right] dt,$$

   where $t_f < \infty$ is fixed, but should be taken as a parameter in your discussion.

6. **Dynamical programming** (10 points)
   You are given the scalar optimal control problem:

$$\dot{x} = -x + u - 1, \quad x(0) = 1,$$

$$J(u) = x^2(1) + \int_0^1 u^2(t)dt,$$

   (a) Obtain the feedback solution to this problem by solving the associated Hamilton-Jacobi-Bellman equation. [Hint: Show that the HJB partial differential equation admits a solution that is a general quadratic function of x, in the form $V(t, x) = \frac{1}{2}p(t)x^2 + k(t)x + m(t)$.]

   (b) Obtain a closed-loop optimal control in the form

$$u(t) = \alpha(t)x(t),$$

   that is, determine the function $\alpha(t)$ above so that u minimizes $J(u)$.