ECE 553, Spring 2008 — Problem Set #4

Issued: March 24th, 2008 — **Due: March 31st, 2008**

Primary Reading: Lecture notes and Handouts.

Homework is due at the beginning of class on the due day!

Problems:

1. **Hamilton-Jacobi and Riccati equations.** (10 points)
Consider the Hamilton-Jacobi equation for the LQ problem studied in class (here I rescaled some of the coefficients to make them 1).

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$

We here complete the proof that the solution $(x(t), \lambda(t))$ to the equation is of the form $\lambda(t) = P(t)x(t)$ with $P(t)$ satisfying the differential Riccati equation (DRE):

$$\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

(a) Consider the "matrix" version of the above Hamilton-Jacobi equation:

$$\begin{bmatrix} \dot{X} \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ \Lambda \end{bmatrix}, \quad X(t_f) = I, \Lambda(t_f) = S.$$

Show that $P(t) = \Lambda(t)X^{-1}(t)$ solves the associated DRE with the proper terminal time condition $P(t_f) = S$. Explain why $P(t)$ is symmetric for all t.

(b) Show that for the solution to the original Hamilton-Jacobi with the given boundary conditions, $\lambda(t)$ and $x(t)$ must be related by $\lambda(t) = P(t)x(t)$. [Hint: you might want to use the uniqueness of solution to differential equations.]

2. **A fourth solution to the same problem.** (5 points)
Using the Lagrangian multiplier method for the end point constraint of the type

$$N(x(t_f), t_f) = 0$$

and the associated transversality conditions given in the handout #2 to solve the problem #4 in Homework #3. Since you know the correct answers, please show clearly all the intermediate steps.

3. **Fixed middle point.** (10 points)
You are given the calculus of variations problem of minimizing

$$J(x) = \int_0^1 \left[\dot{x} - \frac{1}{3}\right]^2 dt \tag{1}$$

over the class piecewise continuously differentiable function $x(\cdot)$, defined on the interval $[0, 1]$, such that $x(0) = 0, x(1) = 1$, and $x(1/2) = 1$.

(a) Obtain a candidate solution (extremal) based on the first-order conditions (and the complete set of boundary conditions). Does the solution you obtained satisfy the strengthened Legendre condition of this problem?

(b) Should we expect the solution above to satisfy the Weierstrass-Erdmann corner conditions at the intermediate point $t = 1/2$?

4. **Limiting behavior of optimal control.** (15 points)
You are given the second-order control system:

$$\dot{x}_1 = x_2, \quad x_1(0) = x_{10},$$
$$\dot{x}_2 = u, \quad x_2(0) = x_{20}$$

where u is the control variable, and t_f is the terminal time which is fixed. We wish to find the control that minimizes the cost functional:

$$J(u) = s_{11}[x_1(t_f)]^2 + s_{22}[x_2(t_f)]^2 + \int_0^{t_f} \left\{[u(t)]^2 + q_{11}[x_1(t)]^2 + q_{22}[x_2(t)]^2\right\} \, dt$$

where $s_{11}, s_{22}, q_{11}, q_{22}$ are non-negative-valued parameters.

(a) Obtain an analytic expression for the optimal feedback controller when $q_{11} = q_{22} = 0$ and t_f is finite, in terms of $s_{11} > 0, s_{22} > 0, t_f > 0$.

(b) Study the limiting behavior of the optimal feedback controller, in each of the three cases: $s_{11} \to 0$, $s_{22} \to 0$, and $t_f \to \infty$. What is the limiting value of the optimum cost in the third case? Any intuitive explanation?

(c) Now solve the infinite-horizon version of the problem, with $q_{11} = q_{22} = 1$, by obtaining the solution to the associated algebraic Riccati equation.

5. **Conjugate point.** (20 points)
Consider the calculus of variations problem of minimizing

$$J(x) = \int_0^{3\pi/2} [\dot{x}^2(t) - x^2(t)] \, dt$$

subject to the end-point restrictions

$$x(0) = 0, \quad x(3\pi/2) = 1.$$

This problem admits the unique extremal: $x^o(t) = -\sin(t)$, which also satisfies the strengthened Legendre condition. But, as we know, this is not a sufficient condition for x^o to be a local minimum. An additional necessary condition is Jacobi's "no conjugate point" condition to be discussed in class on Tuesday.

(a) For the problem above, write down the Jacobi's equation and show that there exists a point in the open interval $(0, 3\pi/2)$ that is conjugate to $t = 0$.

(b) Existence of a conjugate point implies that there exists an admissible variation η defined on this interval around x^o which makes $J(x + \epsilon\eta)$ not larger than $J(x^o)$ for sufficiently small ϵ. Show explicitly that this is indeed the case, by choosing the variation as $\eta(t) = \sin(2t/3)$.