ECE 553, Spring 2008 — Problem Set #1

Issued: February 5, 2008 — **Due: February 13th, 2008**

Primary Reading: Lecture notes, and review analysis, nonlinear optimization, and differential equations.

Homework is due at the beginning of class on the due day!

Problems:

1. **Strong and week neighborhoods** (10 points)
   Consider the family of functions $f_\lambda(t) = \lambda t(1-t)$ defined on $[0,1]$.
   (a) For what values of λ the function f_λ is in the strong $\frac{1}{2}$-neighborhood around 0, *i.e.*$\|f_\lambda - 0\|_{C^1} \le \frac{1}{2}$?
   (b) For what values of λ the function f_λ is in the weak $\frac{1}{2}$-neighborhood around 0, *i.e.*$\|f_\lambda - 0\|_\infty \le \frac{1}{2}$?
   (c) Which of the two neighborhoods is larger, $N_\epsilon^s(0)$ or $N_\epsilon^w(0)$?

2. **Necessary conditions for extreme** (10 points)
   Following what we did in class, try to find the necessary conditions associated to the following minimization problems:
   (a) $\min J(y) = \int_a^b L(y(t), y^{(1)}(t), \ldots, y^{(n)}(t), t)\ dt, \quad y \in C^n[a,b].$
   (b) $\min J(x,y) = \int_a^b L(x(t), \dot{x}(t), y(t), \dot{y}(t))\ dt, \quad x, y \in C^1[a,b].$

   You need to specify what class of variations you use. [Note: You may not necessarily end up with an equation per se but try to simplify your resulting equations as much as possible.]

3. **Minimal surface of revolution** (10 points)
   In class, we derived the functional associated to the minimal surface of revolution problem
   $$\min A(y) = 2\pi \int_a^b y(x)\sqrt{1 + y'(x)^2}\ dx, \quad \text{s.t.} \quad y(a) = y_a, y(b) = y_b.$$
   (a) What is the Euler-Lagrange equation for this problem?
   (b) Solve the resulting ordinary differential equation. [Hint: you may need to use the derivative of the function $\cosh(x)$.]

4. **The necessity of optimality conditions** (20 points)
   Consider the following minimization problem:
   $$\min J(x) = \int_0^\pi x(t)^2[1 - \dot{x}(t)^2]\ dt, \quad \text{s.t.} \quad x(0) = x(\pi) = 0.$$
   (a) Derive the necessary Euler-Lagrange equation that any local extremum needs to satisfy.
   (b) Show that $x^o(t) \equiv 0$ satisfies the equation (including boundary conditions). What is the value of $J(x^o)$?

(c) Consider the function $x_{(n)}(t) = \frac{1}{\sqrt{n}} \sin(nt)$. Show that for any given $\epsilon_o > 0$, there exists n_o such that $x_{(n)} \in N^w_{\epsilon_o}(x^o)$ for all $n > n_o$ and furthermore, $J(x_{(n)}) < 0$ for n sufficiently large.

This concludes that x^o cannot be a *strong* local minimum although it does satisfy the Euler-Lagrange equation. Hence, E-L is only necessary (not sufficient) for local extremum. (Be aware, we however did not show that x^o cannot be a weak minimum.)

5. **Another lemma from Du Bois and Reymond** (10 points)
Let $f(t)$ be a real-valued piecewise continuous function defined on $[a, b]$, and let $\mathbf{N}$ be the set of piecewise continuously differentiable functions defined on the same interval with the boundary conditions $\eta(a) = \eta(b) = 0$. Show that

$$\int_a^b f(t)\dot{\eta}(t)\ dt = 0, \quad \forall \eta \in \mathbf{N},$$

if and only if

$$f(t) \equiv c \quad \text{(a constant)}, \quad \text{for all} \quad t \in [a, b],$$

except possibly at a finite number of points in $[a, b]$. [Hint: since η can be arbitrary, consider choosing $\eta(t) = \int_a^t [c - f(s)]\ ds$ where $c = \frac{1}{b-a} \int_a^b f(s)\ ds$.]

Brain Exercises (optional):

1. Show that the set of continuous functions defined on $[a, b]$, i.e. the space $C^0[a, b]$, is complete under the infinite-norm $\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|$. That is, you need to show that any Cauchy sequence in this space converges to a continuous function on $[a, b]$.

2. Construct an example of a function $J(\cdot)$ that is defined on a bounded closed set D of a function space V but does not have a global maximum or minimum.

3. Generalize the notion of a convex function to a "convex" functional. Convince yourself (rigorously) that a local minimum of a convex functional (defined on a convex domain) is also a global minimum.