# Morse Index Theorem on Riemannian Manifolds *

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## 1 Introduction

*Morse Index Theorem* is considered as one of the most important and fundamental results obtained in Riemannian Geometry. This theorem later turns out to be quite helpful for people to further understand the (homotopy) structure of Riemannian manifolds, and eventually leads to the well-known *Morse Theory*. The primary purpose of this report is to give a concise as well as self-contained introduction to the Morse Index Theorem with minimum requirement of knowledge in Riemannian Geometry.[1]

## 2 Riemannian Geometry Basics

In this section, we review the basic concepts in Riemannian Geometry as well as introduce some notations.

**Definition 1** *A* **Riemannian metric** *on a differential manifold $M$ is given by a inner product (a positive definite bilinear form) $< \cdot, \cdot >$ on the tangent space $T_p(M)$ which depends smoothly on the base point $p$. A* **Riemannian manifold** *is a differential manifold equipped with a Riemannian metric.*[2]

In order to be able to do "differentiation" on the vector bundle over a differentiable manifold, a *connection*, or a *covariant derivative* needs to be defined [3]. We are particularly interested in the covariant derivative of vector fields on Riemannian manifolds. Let us denote the space of all smooth vector fields on a Riemannian manifold $M$ by $\mathcal{X}(M)$, *i.e.* the

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[1]The proof given below essentially follows the lecture notes from Professor Hsiang's course "Riemannian Geometry" given at UC Berkeley in the spring semester 1997.

[2]This definition follows Jost [3].

[3]A general, therefore abstract, definition of connection for a vector bundle over a differentiable manifold can be found in Jost [3]

space of smooth cross-sections of the tangent bundle $T(M)$ of $M$. For any two vector fields $X, Y \in \mathcal{X}(M)$, we use $D_X Y$ to denote the covariant derivative of $Y$ along $X$.

**Definition 2** *A* $C^\infty$ **connection** *$D$ on a differentiable manifold $M$ is a mapping $D : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ denoted by $D : (X, Y) \to D_X Y$ which has the linearity properties: For all $f, g \in C^\infty(M)$ and $X, X', Y, Y' \in \mathcal{X}(M)$, we have* [4]

$$\begin{aligned} D_{fX+gX'}Y &= fD_XY + gD_{X'}Y & (1) \\ D_X(fY + gY') &= fD_XY + gD_XY' + (Xf)Y + (Xg)Y. & (2) \end{aligned}$$

Among all the possible connections on a Riemannian manifold, the *Levi-Civita connection* is of particular interest. The Levi-Civita connection is a *metric* and *torsion-free* connection on the Riemannian manifold (according to Jost [3]). That is, it is a connection which satisfies two further properties:

$$D_XY - D_YX - [X, Y] = 0 \quad (3)$$

$$X < Y, Z > = < D_XY, Z > + < Y, D_XZ > . \quad (4)$$

The equation (4) means the Levi-Civita connection preserves the metric carried by the Riemannian manifold, and the property (3) is called torsion-free. Existence and uniqueness of the Levi-Civita connection is given by the following theorem:

**Theorem 1** *On each Riemannian manifold M, there is precisely one metric and torsion-free connection D and it is given by the formula:*

$$\begin{aligned} < D_XY, Z > &= \frac{1}{2}\{X < Y, Z > - Z < X, Y > + Y < Z, X > \\ &- < X, [Y, Z] > + < Z, [X, Y] > + < Y, [Z, X] >\}. \quad (5) \end{aligned}$$

It is also not hard to show that the Levi-Civita covariant derivative is actually equivalent to the intuitive definition of covariant derivative on a submanifold of Euclidean space $R^n$ by using orthogonal projection (see Boothby [1]).

Given a Riemannian manifold $M$ and its Levi-Civita connection $D$, we therefore can define the *Riemann Curvature Tensor* $R$ on $M$:

**Definition 3** *The* **Riemann Curvature Operator** *$R$ is defined as a multi-linear mapping of $\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that for any $X, Y$ and $Z \in \mathcal{X}(M)$:* [5]

$$R_{XY}Z = -D_XD_YZ + D_YD_XZ + D_{[X,Y]}Z. \quad (6)$$

*Then for $X, Y, Z, W \in \mathcal{X}(M)$, $R(X, Y, Z, W) = < R_{XY}Z, W >$ defines a $C^\infty$-covariant tensor of order 4 which only depends on the Riemannian metric on $M$.*

The Riemann curvature tensor $R$ is a very important geometric object for studying Riemannian Geometry. In particular, the curvature tensor $R$ naturally shows up in the second variational formula of geodesics, which plays an important role in the study of *Jacobi vector fields* and the *Morse index form*, the main topics of this report.

[4] This definition follows Boothby [1].

[5] To be consistent to Professor Hsiang's notations, the Riemann curvature tensor defined here differs from that of Jost [3] or Boothby [1] by a negative sign.

# 3 Geodesics on Riemannian Manifolds

In this section, we study the geometry of *geodesics* on a Riemannian manifold. We first give the definition of geodesics:

**Definition 4** *Given a Riemannian manifold $M$, a (smooth) curve $\gamma : t \in [a,b] \to M$ is called* **geodesic** *if and only if $\gamma(t)$ satisfies*

$$D_{\dot\gamma}\dot\gamma \equiv 0. \tag{7}$$

This definition simply says that the tangent vector $T = \dot\gamma = d\gamma(\frac{\partial}{\partial t})$ is parallel along a geodesic. The concept of geodesic is closely related to the concepts of "shortest arclength" or "least energy" curve on the Riemannian manifold. For a curve $\gamma(t),\ t \in [a,b]$, its arclength $L(\gamma)$ and energy $E(\gamma)$ is, respectively, given by the formula:

$$L(\gamma) = \int_a^b < T, T >^{\frac{1}{2}}\ dt \tag{8}$$

$$E(\gamma) = \frac{1}{2}\int_a^b < T, T >\ dt. \tag{9}$$

## 3.1 First Variational Formula of Arclength and Energy

Now consider a *variation* of a given smooth curve $\gamma(t)$, $t \in [a,b]$ on $M$, *i.e.* a differentiable map

$$\Gamma : (t,u) \in [a,b] \times (-\epsilon, \epsilon) \to M \tag{10}$$

with $\Gamma(\cdot, 0) = \gamma_0(\cdot)$ coincides to the given curve $\gamma$. The variation is called *proper* if the endpoints $\gamma(a)$ and $\gamma(b)$ stay fixed, as shown in Figure 1.

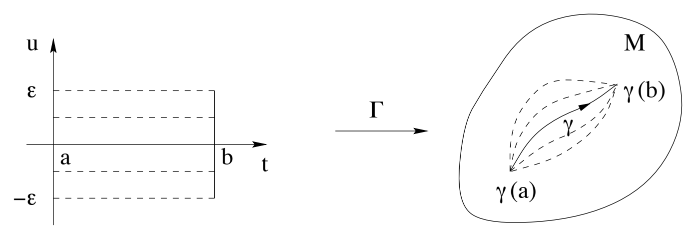


Figure 1: A proper variation of curve $\gamma(t),\ t \in [a,b]$.

To simplify the notations, we define $U = \Gamma_*(\frac{\partial}{\partial u})$ and $T = \Gamma_*(\frac{\partial}{\partial t})$ which correspond to two vector fields on $M$, and we call $U$ the variational vector field. Then, we have the **first variation of the arclength** $L(u)$:

$$\begin{aligned} L'(u) &= \frac{d}{du}\int_a^b < T, T >^{\frac{1}{2}}\ dt \\ &= \int_a^b \frac{1}{\|T\|} < T, D_U T >\ dt \\ &= \int_a^b \frac{1}{\|T\|} < T, D_T U >\ dt, \quad \text{since } D \text{ is torsion-free} \\ &= \int_a^b \frac{1}{\|T\|} (T < T, U > - < D_T T, U >)\ dt, \quad \text{integration by parts} \end{aligned} \tag{11}$$

and the **first variation of the energy** $E(u)$:

$$\begin{aligned} E'(u) &= \frac{1}{2}\frac{d}{du}\int_a^b < T, T > \, dt \\ &= \int_a^b < D_U T, T > \, dt \\ &= \int_a^b < D_T U, T > \, dt \\ &= < T, U > \Big|_a^b - \int_a^b < D_T T, U > \, dt. \end{aligned} \tag{12}$$

For proper variations (that keep the endpoints fixed) $U(a) = U(b) = 0$, it gives $< T, U > \Big|_a^b = 0$. We then have that $E'(0) \equiv 0$ for all proper variations, if and only if

$$D_T T \equiv 0. \tag{13}$$

That means $\gamma$ must be a geodesic. In the special case that $\gamma$ is parameterized proportionally to its arclength[6], *i.e.* $\|T\| \equiv const.$, (13) can also be derived from the first variation of arclength. In this case, (11) becomes

$$L'(0) = \frac{1}{\|T\|}\left( < T, U > \Big|_a^b - \int_a^b < D_T T, U > \, dt \right). \tag{14}$$

Obviously, $L'(0) \equiv 0$ for all proper variations if and only if $D_T T = 0$, *i.e.* $\gamma$ is a geodesic.[7]

### 3.2 Second Variational Formula of Arclength and Energy

We now compute the second derivatives of $E(u)$ and $L(u)$ around a geodesic $\gamma$.

$$\begin{aligned} E''(u) &= \frac{1}{2}\frac{d^2}{du^2}\int_a^b < T, T > \, dt \\ &= \int_a^b U < D_U T, T > \, dt \\ &= \int_a^b U < D_T U, T > \, dt, \quad \text{since } D \text{ is torsion-free} \\ &= \int_a^b (< D_U D_T U, T > + < D_T U, D_U T >) \, dt \\ &= \int_a^b (- < R_{UT} U, T > + < D_T D_U U, T > \\ &\qquad + < D_T U, D_T U >) \, dt, \quad \text{from the definition of } R. \end{aligned} \tag{15}$$

Since $\gamma = \gamma_0$ is a geodesic, *i.e.* $D_T T = 0$, $< D_T D_U U, T >= T < D_U U, T >$. Then we have the **second variation of energy**:

$$E''(0) = < D_U U, T > \Big|_a^b + \int_a^b (< D_T U, D_T U > - < R_{UT} U, T >) \, dt. \tag{16}$$

[6]If a curve $\gamma$ is a geodesic, it must be parameterized proportionally to its arclength. This is because $D_T < T, T >= 2 < D_T T, T >= 0$.

[7]One can actually use these conditions as alternative definitions for geodesics.

$$
\begin{aligned}
L''(u) &= \frac{d^2}{du^2}\int_a^b < T, T >^{\frac{1}{2}}\, dt \\
&= \int_a^b U\left(\frac{1}{\|T\|} < T, D_U T >\right) dt \\
&= \int_a^b \left\{ -\frac{1}{\|T\|^3} < T, D_T U >^2 + \frac{1}{\|T\|} < D_T D_U U, T > \right. \\
&\qquad \left. + \frac{1}{\|T\|}(< D_T U, D_T U > - < R_{UT} U, T >) \right\} dt. \qquad (17)
\end{aligned}
$$

Since $\gamma$ is a geodesic, it is parameterized proportionally to its arclength. Particularly, we may choose the arclength as its parameter, *i.e.* $\|T\| = 1$. For $u = 0$, (17) becomes:

$$
L''(0) = < D_U U, T > \Big|_a^b + \int_a^b (\|D_T U\|^2 - \| < T, D_T U > \|^2 - < R_{UT} U, T >)\, dt. \qquad (18)
$$

Let $U^\perp$ and $D_T U^\perp$, respectively, denote the perpendicular part of the $U$ and $D_T U$ to the tangent vector $T$. Then

$$
\|D_T U\|^2 = \|D_T U^\perp\|^2 + \| < T, D_T U > \|^2 \qquad (19)
$$

$$
< R_{UT} U, T > = < R_{U^\perp T} U^\perp, T >, \quad \text{since } R \text{ is anti-symmetric.} \qquad (20)
$$

We then have the **second variation of arclength**:

$$
L''(0) = < D_U U, T > \Big|_a^b + \int_a^b (\|D_T U^\perp\|^2 - < R_{U^\perp T} U^\perp, T >)\, dt. \qquad (21)
$$

This formula tells us that only the perpendicular part $U^\perp$ of the variational vector field $U$ has contribution to the variation of arclength.

The second variational formula (16) and (21) are very important for the study of the Morse Index Theorem. Notice that the variational formula of arclength need extra constraints on the parameterization of the curve $\gamma$, while the variational formula of energy do not. Therefore in the following, unless explicitly stated, we choose the variational formula of energy to study the Morse Index Theorem.[8]

# 4 Jacobi Vector Fields

In the previous section, we studied general curve variations on Riemannian manifolds. In this section, we restrict our attention to a special class of variations and vector fields associated to them, *i.e.* the *Jacobi variations* and *Jacobi fields*. As we will soon see, the Jacobi fields have very special properties and thus play a central role in the proof of Morse Index Theorem.

[8] Strictly speaking, one can use the variational formula of either energy or arclength to derive Morse Index Theorem. Apparently, most recent Riemannian Geometry books [2] [3] [4] prefer to use the variational formula of energy to prove Morse Index Theorem, instead of using arclength. For an approach of purely using variational formula of arclength, one is referred to Wu [5].

## 4.1 Jacobi Fields and Jacobi Variations

**Definition 5** *Let $\gamma : t \in [a,b] \to M$ be geodesic. A vector field $U$ along $\gamma$ is called* **Jacobi field** *if*

$$D_T D_T U + R_{TU} T = 0 \tag{22}$$

The equation (22) is called the *Jacobi equation* and is usually written as a shorthand by $\ddot{U} + R_{\dot{\gamma}U}\dot{\gamma} = 0$.

**Definition 6** *Let $\gamma : t \in [a,b] \to M$ be geodesic. A variation of $\gamma$, $\Gamma : (t,u) \in [a,b] \times (-\epsilon, \epsilon) \to M$ is called* **Jacobi variation** *if all curves $\Gamma(\cdot, u) = \gamma_u(\cdot)$ are geodesics.*

The following theorem simply shows that there is a one-to-one correspondence between the set of all Jacobi fields and the one of all Jacobi variations.

**Theorem 2** *Let $\gamma : t \in [a,b] \to M$ be geodesic and $\Gamma : (t,u) \in [a,b] \times (-\epsilon, \epsilon) \to M$ is a Jacobi variation of $\gamma$. Then, the vector field*

$$U = \Gamma_*(\frac{\partial}{\partial u})\Big|_{u=0} \tag{23}$$

*is a Jacobi field along $\gamma = \gamma_0$. Conversely, every Jacobi field along $\gamma$ may be obtained by a Jacobi variation of $\gamma$.*

*Proof.* As before, let $T = \Gamma_*(\frac{\partial}{\partial t}), U = \Gamma_*(\frac{\partial}{\partial u})$. Then, $[T,U] = 0$ and $D_T U = D_U T$ because of torsion-free, and $D_T T = 0$ since $\gamma_u$ are all geodesics. We then have

$$\begin{aligned} D_T D_T U &= D_T D_U T \\ &= D_T D_U T - D_U D_T T - D_{[T,U]} T, \quad \text{since } D_T T \text{ and } [T,U] \text{ are } 0 \\ &= -R_{TU} T, \quad \text{from the definition of } R. \end{aligned} \tag{24}$$

Thus, $U$ is a Jocobi field.

Conversely, let $U(t)$ be a Jacobi field along $\gamma(t)$. Let $\xi$ be the geodesic $\xi : u \in (-\epsilon, \epsilon) \to M$ with $\xi(0) = \gamma(0), \xi'(0) = U(0)$. Further, let $T(u)$ and $W(u)$ be parallel vector fields along $\xi$ with

$$T(0) = \dot{\gamma}(0), \quad W(0) = \dot{U}(0). \tag{25}$$

Then one can easily check that

$$\Gamma(t,u) = exp_{\xi(u)}(t(T(u) + uW(u))) \tag{26}$$

gives a Jacobi variation with $U(t)$ as its Jacobi field along $\gamma$. $\square$

## 4.2 Basic Properties of Jacobi Fields

**Lemma 1** *Let $\gamma : [a, b] \to M$ be geodesic. For any $v, w \in T_{\gamma(a)}(M)$, there (locally) exists a unique Jacobi field $U$ along $\gamma$ with*

$$U(a) = v, \quad \dot{U}(a) = w \tag{27}$$

*Proof.* Let $E_1, \ldots, E_n$ be a parallel orthonormal basis of $T(M)$ along $\gamma$. Then any vector field $U$ along $\gamma$ is written as

$$U = u^i E_i \tag{28}$$

Since the vector fields $E_i$ are parallel, we have

$$D_T D_T U = \frac{d^2 u^i}{dt^2} E_i \tag{29}$$

and if we write $R_{TE_i}T = \eta_i^k E_k$ then we have

$$R_{TU}T = u^i \eta_i^k E_k. \tag{30}$$

Then the Jacobi equation (22) now becomes a system of second order ODE

$$\frac{d^2 u^k}{dt^2} + u^i \eta_i^k = 0 \quad k = 1, \ldots, n \tag{31}$$

the solution of which is (locally) uniquely decided by its initial values $U(a)$ and $\dot{U}(a)$. □

Lemma 1 shows that the dimension of Jacobi fields is $2n$ (twice as the dimension of the underlying Riemannian manifold $M$). From the linearity of the Jacobi equation (22), it is not hard to show that linear combinations of Jacobi fields are also Jacobi fields. We then have

**Corollary 1** *The space of all Jacobi fields of a given geodesic $\gamma$ is a $2n$-dimensional vector space.*

We have already known that one can obtain a Jacobi field from a Jacobi variation of geodesics. Now consider for a given geodesic $\gamma([a, b]) \subset M$ the exponential map $exp_{\gamma(a)} : T_{\gamma(a)}(M) \to M$. A rotation of $T_{\gamma(a)}(M)$ about its origin gives a natural Jacobi variation of radial geodesics. Since such a variation always fixes the origin $\gamma(a)$, it corresponds to a Jacobi field $U$ with $U(a) = 0$, as shown in Figure 2.

**Lemma 2** *Let $\gamma : [a, b] \to M$ be a geodesic, $p = \gamma(a)$, i.e.*

$$\gamma(t) = exp_p(t\dot{\gamma}(a)) \tag{32}$$

*For any $w \in T_p M$, the Jacobi field $U(t)$ along $\gamma$ with $U(a) = 0$ and $\dot{U}(a) = w$ is then given by*

$$U(t) = (Dexp_p)(t\dot{\gamma}(a))(tw) \tag{33}$$

*i.e. the derivative of the exponential map $exp_p : T_p(M) \to M$, evaluated at the point $t\dot{\gamma}(a)$ and then applied to the vector $tw$.*

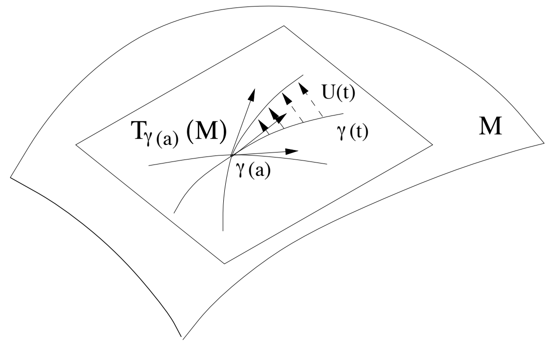


Figure 2: The Jacobi field $U(t)$ associated with rotation of radial geodesics.

*Proof.* One can easily check that the Jacobi field given by the Jacobi variation

$$\Gamma(t,u) = exp_p(t(\dot{\gamma}(a) + uw)) \tag{34}$$

has all the desired properties. □

This lemma shows that the derivative of the exponential map can be computed from Jacobi fields along radial geodesics. Consequently, the critical points of the exponential map is closely related the conjugate points (defined latterly) of geodesics, as we will soon see.

# 5 Morse Index Theorem

With the basic knowledge of the second variation formula and the Jacobi fields, we are now ready to study the Morse Index Theorem.

**Definition 7** *For a geodesic $\gamma : [a,b] \to M$, define $\mathcal{V}$ to be the space of all piecewise $C^\infty$ vector fields along $\gamma$ and let $\mathcal{V}_0 \subset \mathcal{V}$ be the space of vector fields satisfying $V(a) = V(b) = 0$. And the* **Morse index form** *is defined to be a bilinear and symmetric form $I(\cdot,\cdot)$ over the space $\mathcal{V}$ such that any $X, Y \in \mathcal{V}$*

$$I(X,Y) = \int_a^b (< D_T X, D_T Y > - < R_{TX}T, Y >)\, dt. \tag{35}$$

*The* **null space** *$\mathcal{N} \subset \mathcal{V}_0$ of the geodesic $\gamma([a,b])$ is defined as*

$$\mathcal{N} = \{U \in \mathcal{V}_0 : I(U,Y) = 0 \quad \forall Y \in \mathcal{V}_0\}\,. \tag{36}$$

*and the* **nullity** *is defined as the dimension of the null space: $\nu = dim(\mathcal{N})$.*

If $X \in \mathcal{V}_0$ is the variational vector field $U$ in the equation (16), we get $I(X,X) = E''(0)$. This shows a close relation between the study of variational formula and Morse Index Theorem. In fact, the Morse index form may considered as the *Hessian of the functional $E(\gamma)$* defined on the path space of $M$ (see [2]). In that case, the Morse index form turns out to be the Hessian of a 2-parameter variation of the curve $\gamma$.

**Definition 8** *Let $\gamma : t \in [a,b] \to M$ be geodesic. For $t \in [a,b], t \neq a$, $\gamma(a)$ and $\gamma(t)$ are called* **conjugate** *along $\gamma$ if the null space of $\gamma([a,t])$ is not empty, i.e. $\nu_t = dim(\mathcal{N}_{\gamma([a,t])}) \neq 0$.*

## 5.1 Properties of Morse Index Form

**Lemma 3** *A vector field $U \in \mathcal{V}$ is a Jacobi field if and only if $I(U, \mathcal{V}_0) \equiv 0$.*

*Proof.* For any smooth vector field $Y \in \mathcal{V}_0$

$$\begin{aligned} I(U,Y) &= \int_a^b (< D_T U, D_T Y > - < R_{TU}T, Y >)\, dt \\ &= < D_T U, Y > \Big|_a^b - \int_a^b < D_T D_T U + R_{TU}T, Y > \, dt \\ &\quad \text{since } T < D_T U, Y > = < D_T U, D_T Y > + < D_T D_T U, Y > \\ &= -\int_a^b < D_T D_T U + R_{TU}T, Y > \, dt. \end{aligned} \tag{37}$$

Therefore, $I(U,Y) \equiv 0$ for all $Y \in \mathcal{V}_0$ is equivalent to $D_T D_T U + R_{TU}T = 0$, *i.e.* $U$ is a Jacobi vector field.[9] □

This lemma immediately gives us the following corollary which may give an alternative definition of conjugate points (see [3]):

**Corollary 2** *An vector field $U$ is in the null space $\mathcal{N}$ if and only if it is a Jacobi field with boundary values zero. Consequently, $\mathcal{N}$ must be a finite dimensional space.*

According to this corollary, we also have

**Corollary 3** *If $b$ is not a conjugate point of $a$ along the geodesic $\gamma([a,b])$, a Jacobi vector field $U$ of $\gamma([a,b])$ is uniquely determined by its boundary values $U(a)$ and $U(b)$.*

*Proof.* If there are two Jacobi fields $U_1, U_2$ share the same boundary values, then $U' = U_1 - U_2$ is a Jacobi field and belongs to $\mathcal{N}$. Since $\mathcal{N}$ is 0-dimensional ($b$ is not a conjugate point), $U'$ must be 0. □

**Lemma 4** *Let $\gamma([a,b])$ be a geodesic interval containing no conjugate points. Then for all $X \in \mathcal{V}_0$, $I(X,X) > 0$ if $X \neq 0$.*

*Proof.* From Lemma 2, that $\gamma([a,b])$ has no conjugate points implies the exponential map $exp_{\gamma(a)} : T_{\gamma(a)}M \to M$ is locally a diffeomorphism (since the derivative $Dexp_{\gamma(a)}$ should always be nonsingular along $t\dot{\gamma}(a)$ from lemma 2, otherwise, $\mathcal{N}$ would not be empty.). Since $exp_{\gamma(a)}$ is also an isometry, this implies that $\gamma([a,b])$ is locally a shortest path connecting points $\gamma(a)$ and $\gamma(b)$. Thus, for proper variations (with fixing endpoints), the second variation of arclength $L''$ at $\gamma$ is always non-negative. This gives that $I(X,X) \geq 0$ for all $X \in \mathcal{V}_0$.

Now suppose that there exist $X_0 \in \mathcal{V}_0$ such that $I(X_0, X_0) = 0$. Then $\forall\, Y \in \mathcal{V}_0$, we have $X_0 \pm \epsilon Y \in \mathcal{V}_0$ and then

$$I(X_0 \pm \epsilon Y, X_0 \pm \epsilon Y) = I(X_0, X_0) \pm 2\epsilon I(X_0, Y) + \epsilon^2 I(Y,Y) \geq 0 \tag{38}$$

[9] In the case that $Y$ is a piecewise smooth vector field, the proof only need to be slightly modified to become general.

$$
\begin{aligned}
&\Rightarrow \quad \pm 2I(X_0, Y) + \epsilon I(Y, Y) \geq 0 \\
&\Rightarrow \quad I(X_0, Y) = 0.
\end{aligned}
$$

From Lemma 3, $X_0$ is a Jacobi field in $\mathcal{V}_0$ and therefore $X_0 = 0$ since there is no conjugate point along $\gamma([a, b])$. Thus, $I(X, X)$ is strictly positive definite on $\mathcal{V}_0$. □

**Lemma 5** *Let $\gamma([a, b])$ be a geodesic interval containing no conjugate points. If $X, U \in \mathcal{V}$ such that $X(a) = U(a), X(b) = U(b)$ and $U$ is a Jacobi vector fields, then $I(U, U) \leq I(X, X)$ and "=" only when $X = U$.*

*Proof.*

$$
\begin{aligned}
0 &\leq I(U - X, U - X) \\
&= I(U, U) - 2I(U, X) + I(X, X) \\
&= < \dot{U}, U > \Big|_a^b - 2 < \dot{U}, X > \Big|_a^b + I(X, X) \\
&= - < \dot{U}, U > \Big|_a^b + I(X, X) \\
&= -I(U, U) + I(X, X)
\end{aligned}
$$

Obviously, $U - X \in \mathcal{V}_0$. Thus, from Lemma 4, we have $I(U, U) = I(X, X)$ if and only if $U - X = 0$. □

## 5.2 Proof for Morse Index Theorem

We are now ready to prove the well-known Morse Index Theorem. First we need to know what is the Morse index:

**Definition 9** *The* **Morse index** *of a geodesic $\gamma([a, b])$, denoted by $Ind(\gamma([a, b]))$, is the dimension of the largest subspace of $\mathcal{V}_0$*[10] *on which the Morse index form $I(\cdot, \cdot)$ is negative definite.*

For $t \in (a, b)$, denote the nullity of $\gamma([a, t])$ by $\nu_t$. Then the Morse Index Theorem says

**Theorem 3 (Morse Index Theorem)** *Let $\gamma : t \in [a, b] \rightarrow M$ be a geodesic. Then there are at most finitely many points $\gamma(t_i)\, t_i \in (a, b)$ conjugate to $\gamma(a)$ along $\gamma([a, b])$, and*

$$
Ind(\gamma([a, b])) = \sum_i \nu_{t_i} \tag{39}
$$

*Proof.* We prove this theorem by two steps. First we prove that the Morse index $Ind(\gamma)$ is finite and there are only finitely many conjugate points. Second we prove that the equation (39) is true.

*Step One.* From Corollary 2, if $\gamma(t_i)$ is a conjugate point to $\gamma(a)$, then the nullity $\nu_{t_i}$ is finite. We still need to prove that such $t_i$ are also finitely many.

[10]Strictly speaking, the Sobolev completion of $\mathcal{V}_0$ (see [3]).

Consider a subdivision of the interval $[a, b]$ by points $a_0 = a, a_1, \ldots, a_m = b$ which is sufficiently fine such that each piece $\gamma([a_i, a_{i+1}])$ itself contains no conjugate point (such a subdivision always exists since the exponential map $exp$ is locally diffeomorphism). For any vector field $X(t) \in \mathcal{V}_0$, construct a piecewise Jacobi field $U(t)$ with $U(a_i) = X(a_i)$ as shown in Figure 3. According to Lemma 3, a Jacobi field is uniquely determined by its boundary values. Thus the piecewise Jacobi field $U(t)$ constructed here is uniquely determined by $X(a_i)$.

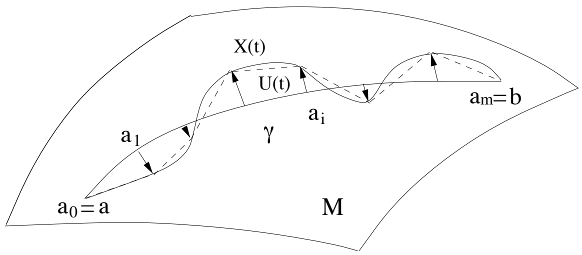


Figure 3: The Jacobi field $U(t)$ associated to an vector $X(t) \in \mathcal{V}_0$ with $U(a_i) = X(a_i)$ where $a_0 = a, a_1, \ldots, a_m = b$ is a subdivision of $[a, b]$ such that each piece $\gamma([a_i, a_{i+1}])$ itself contains no conjugate point.

Then $X'(t) = X(t) - U(t)$ is a vector field with $X'(a_i) = 0$. Since $\gamma([a_i, a_{i+1}])$ has no conjugate point, from Lemma 4, we have $I(X', X') > 0$ if $X' \neq 0$. This means that the space $\mathcal{V}_0$ can be decomposed to two subspace $\mathcal{T}_1$: the subspace of piecewise Jacobi fields $U(t)$ with boundary values $U(a) = U(b) = 0$; and $\mathcal{T}_2$: the subspace of all vector field $X'$ on $\gamma([a, b])$ with $X'(a_i) = 0$. Then

$$\mathcal{V}_0 = \mathcal{T}_1 + \mathcal{T}_2 \tag{40}$$

and the Morse index form $I(\cdot, \cdot)$ is positive definite on $\mathcal{T}_2$.

Then the maximal subspace on which $I(\cdot, \cdot)$ is negative definite is contained in the subspace $\mathcal{T}_1$. For our construction, the dimension of $\mathcal{T}_1$ is $m - 1$ (the number of boundary values $U(a_i)$ needed to specify). Thus the Morse index $Ind(\gamma)$ has to be finite.

Note that $\mathcal{T}_1$ also contains the maximal semi-negative definite subspace of $I(\cdot, \cdot)$. For each point $\gamma(t_i)$ conjugate to $\gamma(a)$, there exists a Jacobi field $U_i(t)$ on the interval $\gamma([a, t_i])$ with $U_i(a) = U_i(t_i) = 0$. Let

$$Y_i = \begin{cases} U_i(t) & \text{for } a \leq t \leq t_i \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

Then $Y_i$ are linearly independent and $I(Y_i, Y_i) = 0$ for all $i$. Thus, there are only finitely many such $Y_i$ since all the $Y_i$ are in $\mathcal{T}_1$. Thus, there are only finitely many points $\gamma(t_i)$ conjugate to $\gamma(a)$ along the geodesic $\gamma([a, b])$.

*Step Two.* In order to prove (39), we now only need to consider $I(\cdot, \cdot)$ as a bilinear form defined on the finite dimensional (vector) space $\mathcal{T}_1$. Define $d_-(t) = Ind(\gamma([a, t]))$. Then $d_-(t)$ is also equal to the number of negative eigenvalues of $I(\cdot, \cdot)$ on $\mathcal{T}_1(\gamma([a, t]))$. As usual, let $\nu_t$ denote the nullity of $\gamma([a, t])$, which is now equal to the number of zero eigenvalues of $I(\cdot, \cdot)$ on $\mathcal{T}_1(\gamma([a, t]))$.

Since eigenvalues of $I(\cdot,\cdot)$ are continuous functions of $t$, when $\nu_t = 0$, for some $\epsilon > 0$

$$d_-(t \pm \epsilon) = d_-(t). \tag{42}$$

When $\nu_t > 0$, *i.e.* there are $\nu_t$ zero eigenvectors $U_i$ of $I(\cdot,\cdot)$, we want to show that $d_-(t+\epsilon) = d_-(t) + \nu_t$. Since positive and negative eigenvalues (locally) stays positive and negative, we only need to study how these zero eigenvalues change.

Suppose $U$ is one of these zero eigenvector, as shown in Figure 4. Then $U$ restricted to

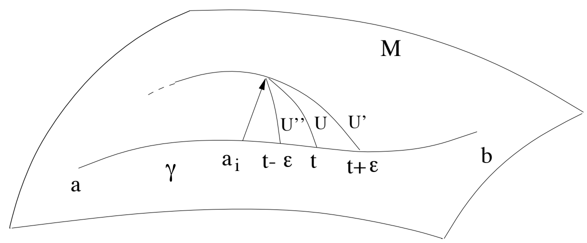


Figure 4: Zero eigenvalues go to negative when $\gamma$ goes through a conjugate point $t$.

the last interval $[a_i, t]$ is a Jacobi vector field with boundary values $U(a_i)$ and $U(t) = 0$. Now let $U'$ be the Jacobi vector field on $[a_i, t+\epsilon]$ with boundary values $U'(a_i) = U(a_i)$ and $U'(t+\epsilon) = 0$ as shown in the Figure 4. Then, according to Lemma 5, on the interval $[a_i, t+\epsilon]$, we have $I(U', U') < I(U, U)$. Now extend $U'$ to $[a, t+\epsilon]$ by making $U' = U$ on $[a, a_i]$. Then we have on $[a, t+\epsilon]$

$$I(U', U') < I(U, U) = 0. \tag{43}$$

Thus the vector field $U'$ is in the negative definite subspace of $I(\cdot,\cdot)$ on $\mathcal{T}_1(\gamma([a, t+\epsilon]))$.[11] The same story happens to all the $\nu_t$ zero eigenvectors $U_i$ of $I(\cdot,\cdot)$ on $\mathcal{T}_1(\gamma([a,t]))$. Since $U_i$ are linearly independent and are also linearly independent to the negative definite subspace of $I(\cdot,\cdot)$ on $\mathcal{T}_1(\gamma([a,t]))$, when the curve go through the conjugate point $t$, the dimension of the maximal subspace of $\mathcal{T}_1(\gamma([a, t+\epsilon]))$ on which $I(\cdot,\cdot)$ is negative definite has to be $d_-(t) + \nu_t$. Thus we get

$$d_-(t+\epsilon) = d_-(t) + \nu_t. \tag{44}$$

Similarly, we also have $d_-(t-\epsilon) = d_-(t) - \nu_t$.

Since there are only finitely many point $\gamma(t_i)$ conjugate to $\gamma(a)$ along $\gamma([a,b])$, we finally have

$$d_-(b) = Ind(\gamma([a,b])) = \sum \nu_{t_i} \tag{45}$$

□

[11] U' is not necessarily a negative eigenvector.

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