

## 4.4 Time-optimal control problems

*Goal:* illustrate the use of the Maximum Principle, but also to reveal fundamental theoretical features of these problems (bang-bang, connections with Lie brackets).

Will go from more specific to more general.

**Example 8** (“Soft landing” [Sussmann, Handout 4, p. 7]). Bring a car to rest at the origin in minimal time using bounded acceleration (*parking problem*).

$$\ddot{x} = u, \quad -1 \leq u \leq 1$$

State-space equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

where  $x_1(0), x_2(0)$  are given. It is clear that the optimal control exists because:

- The system is controllable (well, tricky for  $|u| \leq 1$ , but clear here)  $\Rightarrow$  can hit 0;
- Control is bounded  $\Rightarrow$  cannot do it arbitrarily fast.

So we expect that there exists a control  $u^*$  that achieves the transfer in some optimal time  $t^*$ .

$\longrightarrow$  More formal discussion of existence—later.

Let us use the Maximum Principle to narrow down the candidates.  $L = 1$ ,  $H = p_0 + p_1 x_2 + p_2 u$ ,  $p_0^* = \text{const}$ ,

$$\begin{pmatrix} \dot{p}_1^* \\ \dot{p}_2^* \end{pmatrix} = \begin{pmatrix} -H_{x_1}|_* \\ -H_{x_2}|_* \end{pmatrix} = \begin{pmatrix} 0 \\ -p_1^* \end{pmatrix}$$

So  $p_1^* = \text{const}$ ,  $\dot{p}_2^* = -p_1^* \Rightarrow p_2^*(t) = \underbrace{-p_1^*}_{\text{const}} t + p_2^*(0)$  (linear function of time). By the Hamiltonian

maximization condition:

$$u^*(t) = \text{sgn}(p_2^*(t)) = \begin{cases} 1 & \text{if } p_2^*(t) > 0 \\ -1 & \text{if } p_2^*(t) < 0 \\ ? & \text{if } p_2^*(t) = 0 \end{cases} \quad (4.9)$$

(when  $p_2^* = 0$ ,  $H$  doesn't depend on  $u$ , and so the Maximum Principle gives no info about  $u^*$ ).

If  $p_2^* = 0$  at isolated points, no problem. Can  $p_2^*$  be identically 0 over some time interval? Suppose  $p_2^* \equiv 0$  on some  $[t_1, t_2]$ . Then  $\dot{p}_2^* = -p_1^* \equiv 0$ , so  $(p_1^*, p_2^*) = 0$ . But we saw earlier that this is impossible, because  $H \equiv 0$  (free terminal time) would then imply  $p_0^* = 0$  and the Maximum Principle says that  $p_0^*, p_1^*, p_2^*$  cannot all vanish simultaneously. So, (4.9) defines  $u^*$  uniquely everywhere except when  $p_2^*$  crosses the 0 value.

How many such crossings can we have?  $p_2^*(t)$  is linear  $\Rightarrow$  at most 1!

Optimal control patterns:  $u^* = 1$ , then  $-1$  (full acceleration, followed by full braking). Or the other way around. Or just 1, or just  $-1$ . This depends on the *initial condition*. (Note that if  $x_1(0) < 0$  then we have to go right, and if  $x_1(0) > 0$  then we have to go left—this symmetry accounts for double the number of cases.)

→ The property that  $u^*$  only takes the extreme values  $\pm 1$  is intuitively natural and important. Such controls are called *bang-bang*.

Is this enough to uniquely determine the optimal control?

*Yes.* Let's plot solutions of  $\ddot{x} = u$  for  $u = \pm 1$ .  $u = 1 \Rightarrow \dot{x} = t + a \Rightarrow x = \frac{1}{2}t^2 + at + b$  ( $a, b$  are constants). Easy to see that  $x = \frac{1}{2}\dot{x}^2 + c$ ,  $c$  is a constant dependent on  $a, b$  ( $c = b - a^2/2$ ). This is a parabola in the  $(x, \dot{x})$ -plane. Similarly, for  $u = -1$  we have  $x = -\frac{1}{2}\dot{x}^2 + c$ .

We need to hit 0, and only two among these two families of trajectories do that. (Can view as flowing back from all points on the switching curve, covering  $\mathbb{R}^2$ .) Discuss what the car is doing. Switch once—at the switching curve.

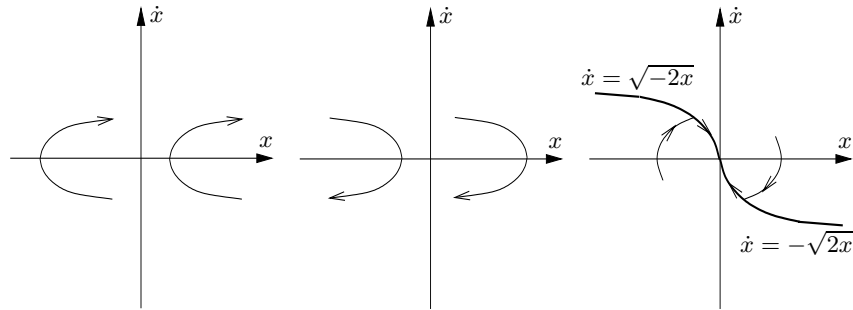


Figure 4.15: Bang-bang time-optimal control of the double integrator: (a) trajectories for  $u = 1$ , (b) trajectories for  $u = -1$ , (c) the switching curve

In this case we have a complete description of  $x^*$ , and  $u^*$  is a *state feedback*. (This is piecewise smooth *regular feedback synthesis*.)

□

For what general classes of systems do we have:

- bang-bang property? – let's see this next;
- Regular feedback synthesis? – hard, won't go into this ([Sussmann's survey]).

**Exercise 14** (due Apr 4) Repeat all the above steps for the modified problem in which we want  $x(t^*) = 0$  but without any constraints on final velocity (“slam into a wall”). Still  $-1 \leq u \leq 1$ , still minimal time problem. Use the Maximum Principle to find the optimal control. Discuss whether it's bang-bang. Address feedback synthesis (describe optimal trajectories in the  $(x, \dot{x})$ -plane).

#### 4.4.1 The bang-bang principle

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m$$

—> LTI, but generalization to LTV is straightforward [Knowles].

*Control set:*

$$\mathcal{U} = \{u \in \mathbb{R}^m : -1 \leq u_i \leq 1, i = 1, \dots, m\}$$

*Hypercube*—natural generalization of the previous example, reasonable (independent constraints on actuators). Can also consider a more general convex set—the bang-bang principle still holds, but the formulas for  $u^*$  are more explicit in the case of hypercube.

*Objective:* steer  $x$  from a given initial state  $x_0$  to a given target state  $x_1$  in *minimal time*.

—> We assume throughout that there exists some control  $u$  that drives  $x_0$  to  $x_1$ .

This is a kind of controllability assumption (but remember that  $\mathcal{U}$  is bounded).

Hamiltonian:  $H = p_0 + p^T(Ax + Bu)$ .

Adjoint equation:  $\dot{p} = -A^T p$ .

the Maximum Principle :  $u^* = \arg \max_{u \in \mathcal{U}} p^T(t)Bu$ .

$$\langle p^*(t), Bu^*(t) \rangle \geq \langle p^*(t), Bu(t) \rangle \quad \forall t, \forall u(t) \in \mathcal{U}$$

or

$$\langle p^*(t), B(u^*(t) - u(t)) \rangle = \sum_{i=1}^n \langle p^*(t), b_i(u_i^*(t) - u_i(t)) \rangle \geq 0$$

where  $b_1, \dots, b_m$  are columns of  $B$ . So, to maximize  $H$  we must have

$$u_i^*(t) = \text{sgn}(\langle p^*(t), b_i \rangle) \quad \forall i$$

(otherwise, can produce  $u(t)$  by changing  $u^*(t)$  in component(s) for which the above formula is false and decrease  $H$ ). We see that  $u_i = \pm 1$  almost everywhere, *unless*  $\langle p^*(t), b_i \rangle \equiv 0$  on some interval of time for some  $i$ .

From  $\dot{p}^* = -A^T p^*$  we have  $p^*(t) = e^{-A^T t} p^*(0) = (e^{-At})^T p^*(0)$ . Thus if  $\langle p^*, b_i \rangle \equiv 0$ , then  $\langle p^*(0), e^{-At} b_i \rangle \equiv 0$ . Differentiating (at  $t = 0$ )—in fact if this is  $\equiv 0$  on some interval then it is 0 everywhere:

$$\langle p^*(0), b_i \rangle = -\langle p^*(0), Ab_i \rangle = \dots = \pm \langle p^*(0), A^{n-1} b_i \rangle = 0$$

i.e.,  $p^*(0)$  is orthogonal to  $b_i, Ab_i, \dots, A^{n-1} b_i$ . As we know,  $p^*(0) \neq 0$  (see Example 6).

—> This will not happen if  $(A, b_i)$  is a controllable pair (controllability with respect to the  $i$ -th input channel).

So, we have the bang-bang property if  $(A, b_i)$  is controllable  $\forall i$ . Such systems are called *normal*. Then  $u^*$  is unique up to measure 0, takes values in the vertices of the hypercube  $\mathcal{U}$ , and actually has a finite number of switches (by analyticity).

#### 4.4.2 Singular optimal controls and Lie brackets

Does the bang-bang principle also hold for nonlinear systems?

*Not always.*

**Example 9**

$$\begin{aligned}\dot{x}_1 &= 1 - x_2^2 \\ \dot{x}_2 &= u\end{aligned}\tag{4.10}$$

$-1 \leq u \leq 1$ . Go from  $(0, 0)$  to  $(1, 0)$  in minimal time.

The optimal control is  $u^*(t) \equiv 0$ .

(Transfer time is 1; any other control would make  $x_2$  deviate from 0, which would slow down the growth of  $x_1$ .)

Note that 0 is an *interior point* of the control set  $[-1, 1] \Rightarrow$  bang-bang principle doesn't hold.

We suspect that some function that determines the sign of  $u^*$  vanishes here—let's take a look.

$H = p_0 + p_1(1 - x_2^2) + p_2u$ .  $\dot{p}_1^* = 0$ ,  $\dot{p}_2^* = 2p_1^*x_2^*$ ,  $u^* = \text{sgn}(p_2^*)$ . Now the canonical equations are more coupled than before.

But the optimal trajectory is contained in  $x_1$ -axis  $\Rightarrow x_2^* \equiv 0 \Rightarrow \dot{p}_2^* \equiv 0$ . So if  $p_2^*(1) = 0$  then  $p_2^* \equiv 0$ . Fixed endpoint  $\Rightarrow$  no constraints on  $p^*(t^*)$ . So, the above optimal control doesn't contradict the Maximum Principle .

□

→ Optimal controls that are not bang-bang are called *singular*. Here we have a *singular arc*.

## Lecture 18

More generally: consider the affine control system

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i, \quad x \in \mathbb{R}^n, \quad -1 \leq u_i \leq 1, \quad i = 1, \dots, m$$

or, in matrix form,

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, & u &\in \mathbb{R}^m \\ J(u) &= \int_{t_0}^{t_1} L(x)dt \end{aligned}$$

Hamiltonian:

$$H = p_0L + \langle p, f + \sum_i g_i u_i \rangle$$

the Maximum Principle :

$$u_i^* = \text{sgn} \langle p^*(t), g_i(x^*(t)) \rangle$$

Call the above inner product  $\varphi_i(t)$ —the *switching function* (always defined along the optimal trajectory  $x^*(t), p^*(t)$ ).

To investigate the bang-bang property, we need to study zeros of the functions  $\varphi_i(t)$ .

→ To simplify calculations, let's assume that  $m = 1$  (only one  $u$ , one  $\varphi$ ) and  $L \equiv 1$  (the *time-optimal problem* as before).

$$\varphi(t) = \langle p^*(t), g(x^*(t)) \rangle$$

$$\begin{aligned} \dot{x}^* &= f + gu^* \\ \dot{p}^* &= -H_x|_* = -(f_x)^T|_* p^* - (g_x)^T|_* p^* u^* \end{aligned}$$

We'll omit  $|_*$ , understood.

Let's compute the derivative of  $\varphi(t)$ .

$$\begin{aligned} \dot{\varphi} &= \langle \dot{p}^*, g \rangle + \langle p^*, g_x \cdot (f + gu^*) \rangle \\ &= - \left\langle (f_x)^T p^*, g \right\rangle - \underbrace{\left\langle (g_x)^T p^* u^*, g \right\rangle}_{= \langle p^*, g_x g \rangle u^*} + \langle p^*, g_x f \rangle + \langle p^*, g_x g \rangle u^* \\ &= \langle p^*, g_x f - f_x g \rangle \end{aligned}$$

$g_x f - f_x g$  is a new vector field, interesting—what is it?

**Example 10**  $f$  and  $g$  are linear vector fields;  $f(x) = Ax$ ,  $g(x) = Bx$ . This corresponds to a *bilinear* (not linear!) system  $\dot{x} = Ax + Bxu$ . We have  $f_x = A$ ,  $g_x = B$ ,

$$g_x f - f_x g = BAx - ABx = (BA - AB)x = [B, A]x$$

$[B, A]$  is *commutator*, or *Lie bracket*.

□

A *Lie bracket* of general vector fields is defined as

$$[f, g](x) := g_x(x)f(x) - f_x(x)g(x)$$

(here  $g_x$  and  $f_x$  are matrices).

→ Note the sign difference.

Meaning of Lie bracket (justification of the term “commutator”):

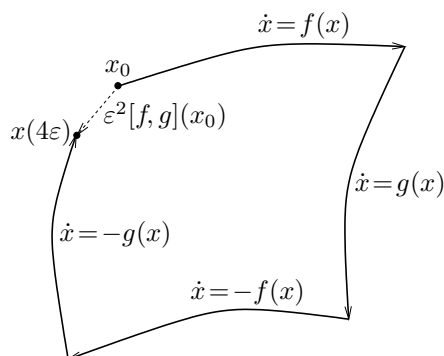


Figure 4.16: Geometric interpretation of the Lie bracket

In particular, if  $f$  and  $g$  commute (i.e., we come back to  $x_0$ ) then  $[f, g] = 0$ .

The equation

$$\dot{\varphi}(t) = \langle p^*(t), [f, g](x^*(t)) \rangle$$

reveals a fundamental relation between Lie brackets and optimal control (read Sussmann). Recall:

$$\varphi(t) = \langle p^*(t), g(x^*(t)) \rangle$$

So to have a singular optimal trajectory, we must have

$$\begin{aligned} \varphi \equiv 0 &\Rightarrow p^*(t) \perp g(x^*(t)) \\ \dot{\varphi} \equiv 0 &\Rightarrow p^*(t) \perp [f, g](x^*(t)) \end{aligned}$$

→  $\dot{\varphi}$  has terms  $[g_i, g_j]$  in the multiple input case.

But higher derivatives of  $\varphi$  must also vanish.

What is  $\ddot{\varphi}$ ? (Ask them to guess.)

Rather than differentiate  $\dot{\varphi}$  again, let's examine our derivation of  $\dot{\varphi}$ . For any function  $h(x)$ , we can show by the same calculation that

$$\frac{d}{dt} \langle p^*, h \rangle = \langle p^*, [f, h] \rangle + \langle p^*, [g, h] \rangle u$$

We had  $h = g$  so  $[g, h]$  was 0. This time we have  $h = [f, g]$ , so

$$\ddot{\varphi} = \langle p^*, [f, [f, g]](x^*) \rangle + \langle p^*, [g, [f, g]](x^*) \rangle u$$

(iterated Lie brackets!) If  $\varphi \equiv 0$ , then  $\langle p^*, g(x^*) \rangle = 0$ ,  $\langle p^*, [f, g](x^*) \rangle = 0$ ,  $\langle p^*, [f, [f, g]](x^*) \rangle + \langle p^*, [g, [f, g]](x^*) \rangle u = 0$ , and so on.

→ Recall that  $p^*(t) \neq 0$  for free-time problems with  $L \neq 0$ .

We see that for  $n = 2$ , we can rule out singularity if  $g$  and  $[f, g]$  are linearly independent along the optimal trajectory. If this is not the case, then the first two equations can hold, and then

$$u^* = -\frac{\langle p^*, [f, [f, g]](x^*) \rangle}{\langle p^*, [g, [f, g]](x^*) \rangle}$$

is potentially a singular optimal control. However, it should meet control constraints. If the constraint is  $|u| \leq 1$ , and if we assume, e.g., that

$$[g, [f, g]](x) = \alpha(x)g(x) + \beta(x)[f, g](x) + \gamma(x)[f, [f, g]](x) \quad \forall x$$

where  $|\gamma(x)| < 1$ , then the above  $u$  would not be admissible.

But then we have to consider the case where third-order brackets vanish and look at  $\ddot{\varphi}$ , and so on. This is the game one plays to derive conditions guaranteeing bang-bang property [Sussmann].

Do Lie brackets explain all results we derived earlier (before discussing Lie brackets)?

- *Linear systems:*

$f = Ax, g = b$  (single input).  $[f, g] = -Ab, [f, [f, g]] = A^2b, [g, [f, g]] = 0, [f, [f, [f, g]]] = -A^3b$ . If  $\text{rank}(b, Ab, \dots, A^{n-1}b) = n$  then  $\varphi$  cannot be  $\equiv 0$ , so controllability comes out and confirms our earlier result.

- *Singular example:*

System (4.10).

$$f = \begin{pmatrix} 1-x_2^2 \\ 0 \end{pmatrix}, g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$f_x = \begin{pmatrix} 0 & -2x_2 \\ 0 & 0 \end{pmatrix} \Rightarrow [f, g] = \begin{pmatrix} 0 & 2x_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 0 \end{pmatrix}$$

We see that on the  $x_1$ -axis ( $x_2 = 0$ ),  $g$  and  $[f, g]$  are *not* linearly independent! So need to look at higher-order brackets.

Next, let's calculate  $[f, [f, g]]$ . *We didn't do this in class.*

$$\frac{\partial}{\partial x}[f, g] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$[f, [f, g]] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-x_2^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & -2x_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}!$$

If  $u \equiv 0$ , then the formula for  $\ddot{\varphi}$  implies  $\ddot{\varphi} \equiv 0$ . All subsequent brackets are 0  $\Rightarrow$  we are done,  $\varphi$  is indeed  $\equiv 0$ .

→ So, Lie brackets indeed contain all information about singularities.

Singular controls are not necessarily complex— $u \equiv 0$  in the example is quite simple.

→ For time-optimal problems in the plane, for  $\dot{x} = f + gu$ , with  $-1 \leq u \leq 1$ , can show that all optimal trajectories are concatenations of a *finite* number of “bang” trajectories ( $u = 1$  or  $u = -1$ ) and “nice” (analytic) singular arcs.

Is this true in general (for  $n > 2$ , or for not necessarily time-optimal problems, even in  $\mathbb{R}^2$ )?

*Example: Fuller's problem*

Studied by Fuller and others since 1960 [Jurdjevic], [Ryan, IJC, 1987], [Sussmann's survey].

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u\end{aligned}$$

As before,  $-1 \leq u \leq 1$ . *Goal:* bring  $x$  from a given  $x_0$  to the origin  $(0, 0)$  and minimize the cost

$$J = \int_{t_0}^{t_f} x_1^2(t) dt, \quad t_f \text{ free}$$

( $t_f$  can be fixed and large enough [Jurdjevic]—doesn't really matter since we can rest at 0 once we get there).

(If we had  $J = \int x_2^2$  then we would have no Fuller phenomenon [Jurdjevic].)

$$H = p_0 x_1^2 + p_1 x_2 + p_2 u$$

the Maximum Principle :  $u^* = \text{sgn}(p_2^*)$  as before.

Adjoint equation (normalizing  $p_0$  to  $-1$ ):

$$\begin{aligned}\dot{p}_1^* &= 2x_1^* \\ \dot{p}_2^* &= -p_1^*\end{aligned}$$

so things are more complicated than before (e.g., in the soft landing example?) because  $x^*$  and  $p^*$  are coupled.

To have a singular trajectory we must have

$$p_2 \equiv 0 \Rightarrow p_1 \equiv 0 \Rightarrow x_1 \equiv 0 \Rightarrow x_2 \equiv 0$$

(trivial).

Let's then look at possible bang-bang trajectories:  $u = \pm 1$ . Integrating:  $x_2 = \pm t + a$ ,  $x_1 = \pm \frac{1}{2}t^2 + at + b$ ,

$$\begin{aligned}p_1 &= \pm \frac{1}{3}t^3 + at^2 + 2bt + c \\ p_2 &= \mp \frac{1}{12}t^4 - \frac{1}{3}at^3 - bt^2 - ct - d\end{aligned}$$

where  $a, b, c, d$  are constants. Switching takes place on the surface  $\{p_2 = 0\}$ .

—> The above formulas imply the following interesting fact: there is no solution that goes from this switching surface to  $(x_1, x_2, p_1, p_2) = (0, 0, 0, 0)$  in any time  $t > 0$ .

So, bang-bang controls are also excluded. And if we are at  $\{p_2 = 0\}$ , then to get a singular trajectory we must first hit  $(0, 0, 0, 0)$ , and no constant controls in *finite* concatenation can achieve this!

So what actually happens?

The only answer is that we have an infinite number of switches, with an accumulation point. This is called *Zeno behavior*.



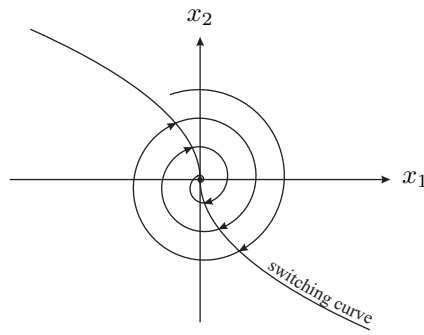


Figure 4.17: Optimal trajectory for Fuller's problem

Infinite number of switches in finite time! Switching intervals decrease in geometric progression. It turns out that such a switching strategy can reach the origin in finite time, if switchings occur on the right curve. Switching curve:  $\{x_1 + \gamma|x_2|x_2 = 0\}$ ,  $\gamma \approx 0.445$ .

Note that his behavior is somewhat related to the time-optimal problem for the same double integrator, but the nature of switching is drastically different (there, we have  $\gamma = 1/2$ ).

*Relation of Fuller's problem to time-optimal problems:*

- Can consider the parameterized class of problems  $\ddot{x} = u$ ,  $J = \int_{t_0}^{t_f} |x|^\nu dt$ ,  $\nu$  is a positive real number. Goal—same: transfer  $x_0$  to 0, minimize  $J$ . For  $\nu = 0$  we recover the time-optimal case. For  $\nu = 2$  we recover Fuller's problem. It turns out that there exists a value  $\bar{\nu} \approx 0.35$  (“bifurcation value”) such that:
  - For  $\nu \in [0, \bar{\nu}]$  the optimal control is bang-bang with at most 1 discontinuity (switch);
  - For  $\nu > \bar{\nu}$  we have Zeno behavior (Fuller's phenomenon).

**Exercise 15** (due Apr 4) Give an example of a system of the form

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad n > 2, \quad -1 \leq u \leq 1$$

such that the problem of transferring a given initial state  $x_0$  to 0 in minimal time has a solution  $u^*$  which involves infinitely many switchings between  $u = 1$  and  $u = -1$ . Hint: base this on Fuller's example.

—> This shows that Theorem 8.4.1 from Sussmann's survey cannot be extended to  $\mathbb{R}^3$ .

Fuller's phenomenon is observed in other problems too, e.g., Dubin's car [Agrachev-Sachkov].

## 4.5 Existence of optimal controls

Let's now examine an issue that we've been dodging for a while.

*Perron's paradox:*

Reference: [Young]

Let  $n$  be the largest positive integer.

*Claim:*  $n = 1$ .

PROOF. Otherwise,  $n^2 > n$ , a contradiction.  $\square$

This is silly, of course, because the largest positive integer doesn't exist. But what does this paradox mean in our context?

Finding the largest positive integer is an optimization problem. The above argument says:  $n$  is optimal  $\Rightarrow n = 1$ . I.e.,  $n = 1$  is a *necessary condition* for optimality! (*So is the Maximum Principle*.) Thus a necessary condition can be useless—even misleading—*unless we know the solution exists*.

Assume we have some optimal control problem, with system  $\dot{x} = f(x, u)$ , cost  $J(u)$ , and target set  $S \subset \mathbb{R} \times \mathbb{R}^n$ . How can we ensure the existence of an optimal control? First, we must obviously assume that there exists at least one control  $u$  that drives  $x$  from  $x_0$  at  $t_0$  to the target set—otherwise the problem is ill-posed.

— This is a kind of *controllability* assumption (nontrivial to check, especially for bounded  $\mathcal{U}$ ), but we usually tacitly assume it.

Is this enough? No!

**Example 11**  $\dot{x} = u$ ,  $x, u \in \mathbb{R}$ , problem: transfer  $x$  from  $x_0 = 0$  to  $x_1 = 1$  in minimal time.

What is  $u^*$ ? Doesn't exist:  $t^* = 0$  is impossible, but any  $t > 0$  can be achieved.  $\square$

In the above example, the problem was that the control set  $\mathcal{U}$  was unbounded, which led to arbitrarily fast transfer.

Would *boundedness* of  $\mathcal{U}$  be enough to fix it?

**Example 12** Same system, same goal, but  $u \in \mathcal{U} := [0, 1]$ .

What is  $t^*$ ?  $t^* = 1$  cannot be achieved, but can transfer from 0 to 1 in any time  $t > 1 \Rightarrow$  again, no optimal solution.  $\square$

So if  $\mathcal{U}$  is not *closed* we still have a problem. Beginning to guess that we need both closedness and boundedness (*compactness*).

— But note that existence of optimal controls has to do not just with  $\mathcal{U}$ , but with the state trajectories produced by controls in  $\mathcal{U}$ . It turns out that this issue is closely related to *reachable sets*.

## Lecture 19

Denote by  $R^t(x_0)$  the set of states reachable from  $x_0$  using all admissible  $u$  ( $t$  is an arbitrary fixed time). The solution of an optimal control problem is related to reachable sets—we saw this in the proof of the Maximum Principle :  $y^*(t^*)$  must be on the boundary of  $R^t(y_0)$ , because if it's in the interior then we can decrease the cost:

(Though in the proof we didn't work with  $R^{t^*}$  directly, but with infinitesimal directions—and the reason is that  $R^{t^*}$  can be complicated.)

And this boundary point must belong to  $R^t$ ! (More accurate phrasing: it should be a boundary point of  $R^t$ .)

Or consider a time-optimal problem:

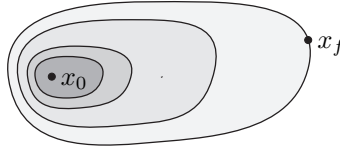


Figure 4.18: Propagation of reachable sets  $R^t(x_0)$

$x_f$  is a boundary point of  $R^{t^*}(x_0)$ —otherwise we could reach it sooner. (See web handout on bang-bang principle.)

We can interpret the above two examples in terms of this property:

- In Example 11,  $R^t(x_0)$  is the entire  $\mathbb{R}$  (*not bounded*)  $\forall t \Rightarrow x = 1$  couldn't be on the boundary for any  $t$ .
- In Example 12,  $R^1(x_0) = [0, 1)$ , and  $x = 1$  is outside (boundary not included in  $R^1(x_0)$ ), so we couldn't reach it. And for any  $t > 1$ ,  $x = 1$  is already in the interior of  $R^t(x_0)$ .  $R^1(x_0)$  is *not closed*.

(Can generalize this discussion to target sets instead of fixed terminal points, even moving target sets.)

So, a refined guess is that for existence of optimal controls, we should have compactness of the reachable set (rather than control set). As we said, reachable sets can be quite complicated—but there exists a general result:

**Theorem 8 (Filippov)** *Given  $\dot{x} = f(t, x, u)$ , assume:*

- *On a given interval  $[t_0, t_f]$ , solutions exist  $\forall u$ ;*
- *$\forall (t, x)$ , the set  $\{f(t, x, u) : u \in \mathcal{U}\}$  is compact and convex.*

*Then  $R^t(x_0)$  is compact  $\forall t \in [t_0, t_f]$ .*

Note: the 1st condition is not implied by the 2nd. Example:  $\dot{x} = x^2 + u$ .

We give no proof but make some comments:

- Filippov's theorem actually makes a stronger statement (from which compactness of  $R^t$  follows): the set of all *trajectories* of the system, as a subset of  $C([t_0, t_f], \mathbb{R}^n)$ , is compact in topology induced by the norm  $\|\cdot\|_0$ , i.e., topology of uniform convergence. (Compact  $\Rightarrow$  closed and bounded  $\Rightarrow$  for each  $t$  its "slice" is also closed and bounded—not hard to prove. Or use sequential compactness.)
- Convexity is not necessary to have boundedness of  $R^t$  (though it's not easy to prove boundedness without convexity [Lin-Sontag-Wang]), but convexity is crucial for closedness of  $R^t$ . The argument showing closedness relies on separation property of convex sets ([Bressan's online lecture notes], on class website).

Begin optional: \_\_\_\_\_

We can see the role of the convexity assumption from the following *example*:

$$\dot{x} = u, \quad u \in \{-1, 1\}$$

The trajectory  $x \equiv 0$  is the uniform limit of a sequence of system trajectories, but it is not a valid trajectory of the system  $\Rightarrow$  the set of trajectories is not closed.

Here  $R^t(x_0)$  are actually compact, but we can build a better example:

$$\begin{aligned}\dot{x}_1 &= u \\ \dot{x}_2 &= x_1^2\end{aligned}$$

$u \in \{-1, 1\}$ ,  $x_0 = (0, 0)$ . Using the above controls, we can reach points arbitrarily close to  $(0, 0)$ , of the form  $(0, \varepsilon)$ , but  $(0, 0) \notin R^t$  because  $\varepsilon > 0$  always ( $x_1$  cannot remain at 0).

If we "convexify" the problem by letting  $\mathcal{U} := \text{co}\{-1, 1\} = [-1, 1]$ , then we get a larger reachable set which will now be closed by the above theorem.

$\rightarrow$  In fact, this larger reachable set will be exactly the closure of the original one, i.e., the original one is dense in the new one.

This is a special case of the general *relaxation theorem* about density of the set of solutions of the original system in the set of solutions of the convexified (relaxed) system.

$\rightarrow$  For affine systems, can see Filippov's Theorem more directly:  $L_{\mathcal{U}}[0, T]$  is weakly compact, and the map  $u \mapsto x(T)$  is continuous with respect to strong topology for  $x \Rightarrow R^t(x_0)$  is compact as an image of compact set  $\mathcal{U}$  [Sontag].

End optional \_\_\_\_\_

- Filippov's Theorem applies to useful classes of systems:

### Nonlinear systems affine in controls

$$\dot{x} = f(x) + G(x)u$$

$u \in \mathcal{U}$ —compact, convex. For each  $x$ , the set  $\{f + Gu : u \in \mathcal{U}\}$  is the image of  $\mathcal{U}$  under an affine map  $\Rightarrow$  it's compact and convex. But need to assume existence of solutions on the time interval we work on!  $\dot{x} = x^2 + u$ ,  $u \in [-1, 1]$  blows up in finite time!

## Linear systems

$$\dot{x} = Ax + Bu$$

(Special case—can be LTV too.)

→ In fact, in this case  $R^t(x_0)$  is also convex—can easily show using the variations-of-constants formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$$

(preserves convex combinations).

OK, how can we use Filippov's Theorem to show existence of optimal controls?

The simplest case is Mayer problem (terminal cost only). Assume that:

- $\dot{x} = f(x, u)$  satisfies the conditions of Filippov's Theorem (always true for linear systems and forward complete affine nonlinear systems);
- Cost is  $J = K(x(t_f))$ , where  $K$  is a continuous function,  $t_f$ -fixed;

Then the problem admits an optimal solution.

Reason: Weierstrass Theorem!

$K$  is a continuous function on a compact set  $R^{t_f}(x_0) \Rightarrow$  it has a global minimum (or maximum).

For more general problems, with free terminal time and running cost, we need to work harder (know that the set of trajectories is compact but don't know if  $J$  is a continuous functional on it). However, there exist useful classes of problems where compactness (actually, below we'll just use closedness) of reachable sets implies existence of optimal controls. Let's see this for *linear time-optimal problems*:

$$\dot{x} = Ax + Bu$$

$u \in \mathcal{U}$ -compact and convex,  $x_0, x_f$ -given, problem: *transfer in minimal time*.

**Theorem 9** Assume  $x_f \in R^t(x_0)$  for some  $t$ . Then there exists a time-optimal control.

PROOF. [Sontag], [Knowles] (same argument in both) By assumption, the set  $\{t : x_f \in R^t(x_0)\}$  is nonempty and bounded from below by 0 (actually it's bounded away from 0 provided  $x_f \neq x_0$ ). We also know that this set is nonempty. So,  $t^* := \inf\{t : x_f \in R^t(x_0)\}$  is well defined.

We'll be done if we show that  $x_f \in R^{t^*}(x_0)$ , i.e., that  $t^*$  is actually a minimum. This will mean that there exists a control  $u^*$  (bang-bang, by the way, if  $\mathcal{U}$  is a *cube* or another convex polyhedron) that transfers  $x_0$  to  $x_f$  in time  $t^*$ , and by definition of  $t^*$  no control does it faster.

(Recall Example 12 at start of lecture: could reach in any time  $t > 1$  but couldn't reach in  $t = 1$ ; here it doesn't happen.)

By definition of infimum, there exists a sequence  $t_k \searrow t^*$  such that  $x_f \in R^{t_k}(x_0) \forall k$ . Each  $k$  satisfies  $x_f \in R^{t_k}(x_0) \Rightarrow$  for some control  $u_k$ , we have

$$x_f = e^{At_k}x_0 + \int_0^{t_k} e^{A(t_k-s)}Bu_k(s)ds$$

→ We want to use closedness of reachable sets, but can't work with  $R^{t_k}$  for different  $k$ —let's truncate:

$$x_f^k := e^{At^*} x_0 + \int_0^{t^*} e^{A(t^*-s)} B u_k(s) ds$$

→ These are different points now—obtained by  $u_k$  acting on a truncated interval  $[0, t^*]$ —but they all belong to the same set  $R^{t^*}(x_0)$ .

*Claim:*  $x_f^k \xrightarrow{k \rightarrow \infty} x_f$  (with respect to Euclidean norm on  $\mathbb{R}^n$ ).

In view of the claim and the fact that  $R^{t^*}(x_0)$  is closed, we have  $x_f \in R^{t^*}(x_0)$  as needed. □

**Exercise 16** (due Apr 11) Prove the claim.

Existence results such as the above theorem are not constructive, but they justify the application of the Maximum Principle which, as we have seen, often allows one to actually find the optimal control.