Solutions:

**Homework #5, Problem 5. (Invariant Distributions).** Let $\Delta$ be a distribution on $\mathbb{R}^n$. This means that for each $x \in \mathbb{R}^n$, $\Delta(x)$ is a subspace of $\mathbb{R}^n$. Let $f$ be a globally Lipschitz, bounded vector field on $\mathbb{R}^n$ with flow $\phi_t$. Assume that $f$ and $\Delta$ are analytic (think about how this can be defined precisely; recall that a function is analytic if it is equal to its Taylor series). We say that $\Delta$ is invariant with respect to $f$ (or $\phi_t$) if

$$(D\phi_t)(\Delta(x)) = \Delta(\phi_t(x)),$$

for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$. (The derivative $D\phi_t$ is taken at the appropriate point, i.e., $x$.)

Show that $\Delta$ is invariant with respect to $f$ if and only if $[f, g] \in \Delta$ for every smooth vector field $g$ such that $g(x) \in \Delta(x)$, for all $x \in \mathbb{R}^n$.

**Proof.** $(\Rightarrow)$ Assume $\Delta$ is invariant relative to $f$, and let $g$ be an analytic vector field in $\Delta$ (i.e. $g(x) \in \Delta(x)$ for all $x$). Then, by invariance,

$$((\phi_{-t})_* g)(x) = D\phi_{-t}(g(\phi_t x)) \in \Delta(x),$$

for all $x$ and $t$. Since $g \in \Delta$ and $\Delta(x)$ is a closed set, we obtain:

$$[f, g](x) = \lim_{t \to 0} \frac{D\phi_{-t}(g(\phi_t x)) - g(x)}{t} \in \Delta(x),$$

as desired.

$(\Leftarrow)$ Assume $[f, \Delta] \subset \Delta$, and let $g$ be an arbitrary analytic vector field in $\Delta$. Consider $D\phi_{-t}(g(\phi_t x))$. It can be verified (using the definition of the Lie bracket) that

$$\left(\frac{d}{dt}\right)^k \bigg|_0 D\phi_{-t}(g(\phi_t x)) = \text{ad}_f^k g(x) = [f, [f, \cdots, [f, g] \cdots]],$$

which is in $\Delta(x)$, by induction. Since the function $t \mapsto D\phi_{-t}(g(\phi_t x))$ is analytic (being composed of analytic functions), it follows that, for small $t$, it is equal to its Taylor series at 0:

$$D\phi_{-t}(g(\phi_t x)) = g(x) + t[f, g](x) + \frac{t^2}{2} [f, [f, g]](x) + \cdots.$$

Each term in this series is in $\Delta(x)$ and $\Delta(x)$ is a closed set, so the sum of the series, namely $D\phi_{-t}(g(\phi_t x))$, is in $\Delta(x)$ as well. If $t$ is large, we can use the semigroup property of the flow and concatenate the preceding argument to prove invariance for any $t$. Thus, $D\phi_{-t}(\Delta(\phi_t x)) \subset \Delta(x)$. In fact, since $D\phi_{-t}$ is a linear isomorphism, $D\phi_{-t}(\Delta(\phi_t x))$ and $\Delta(x)$ have the same dimension, so they must be equal. A simple change of variables: $s = -t, y = \phi_t x$, yields $D\phi_s(\Delta(y)) = \Delta(\phi_s y)$, for all $y$ and $s$.

**Note.** This result allows us to check invariance of $\Delta$, the definition of requires the knowledge of the flow of $f$, using only Lie brackets which can be computed (sometimes by hand, sometimes by a computer).
1. (Smooth Stabilizability). Assume that the control system

\[ \dot{x} = f(x, u) \]

is smoothly exponentially stabilizable at \( 0 \in \mathbb{R}^n \), that is, there exists a smooth control law \( u = k(x) \) such that \( 0 \) is a locally exponentially stable equilibrium of the closed loop system \( \dot{x} = f(x, k(x)) \). Prove that the extended system

\[
\begin{align*}
\dot{x} &= f(x, z) \\
\dot{z} &= h(x, z) + u,
\end{align*}
\]

where \( h \) is a smooth function, is also exponentially stabilizable by smooth feedback. (Hint: Try a Lyapunov function \( W(x, z) = V(x) + \frac{1}{2}|k(x)|^2 \), where \( V \) is an appropriate Lyapunov function for the original system.)

**Proof.** Since the origin is a locally exponentially stable equilibrium for \( \dot{x} = f(x, k(x)) \), it follows that there exists a Lyapunov function \( V \) such that

\[
\alpha_1|x|^2 \leq V(x) \leq \alpha_2|x|^2, \quad \nabla V(x) \cdot f(x, k(x)) \leq -\alpha_3|x|^2, \quad |\nabla V(x)| \leq \alpha_4|x|, \tag{1}
\]

for all \( x \) in a neighborhood of the origin. Without loss, we can assume \( k(0) = 0 \). Smoothness of \( f \) and the Fundamental Theorem of Calculus imply that there exists a matrix valued function \( G(x, z) \) such that

\[ f(x, k(x) + z) = f(x, k(x)) + G(x, z)z, \]

or equivalently

\[ f(x, z) = f(x, k(x)) + G(x, z - k(x))(z - k(x)). \]

Let

\[ W(x, z) = V(x) + \frac{1}{2}|z - k(x)|^2, \]

and define \( w = z - k(x) \). Instead of the \((x, z)\)-system, consider the \((x, w)\)-system:

\[
\begin{align*}
\dot{x} &= f(x, k(x) + w) \\
\dot{w} &= g(x, w),
\end{align*}
\]

where \( g(x, w) = h(x, k(x) + w) + u - \nabla k(x) \cdot f(x, k(x) + w) \). Observe that the \((x, z)\)- and \((x, w)\)-systems are equivalent: if \((x(t), z(t))\) is an orbit of the \((x, z)\)-system, then \((x(t), z(t) - k(x(t)))\) is an orbit of the \((x, w)\)-system. Conversely, if \((x(t), w(t))\) is an orbit of the \((x, w)\)-system, then \((x(t), w(t) + k(x(t)))\) is an orbit of the \((x, z)\)-system.

It is not hard to see that \( W \), as a function of \( x \) and \( w \), satisfies the analogues of the first and third conditions in (1). Define a feedback law by

\[ u = K(x, z) = -h(x, z) + \nabla k(x) \cdot f(x, z) - [\nabla V(x) G(x, z - k(x))]^T - z + k(x). \]

The orbital derivative \( \dot{W} \) of \( W \) relative to the closed loop \((x, z)\)-system is

\[
\nabla V(x) \cdot f(x, z) + [h(x, z) + K(x, z) - \nabla k(x) \cdot f(x, z)]^T (z - k(x)),
\]

which reduces to

\[ \dot{W} = \nabla V(x) \cdot f(x, k(x)) - |z - k(x)|^2. \]
Note that the orbital derivative of $W$ relative to the $(x, w)$-system is the same as that relative to the $(x, z)$-system.

As a function of $(x, w)$, $-\dot{W}$ is locally positive definite. Therefore, $W$ is a Lyapunov function for the closed loop $(x, w)$-system (with feedback $u = K(x, z) = K(x, k(x) + w)$), which proves that it has a locally exponentially stable equilibrium at the origin. That is, if $(x(0), w(0))$ is close enough to $(0, 0)$, then $(x(t), w(t)) \to (0, 0)$, as $t \to \infty$, exponentially fast. Therefore, $(x(t), z(t)) = (x(t), w(t) + k(x(t))) \to (0, k(0)) = (0, 0)$, as $t \to \infty$, also exponentially fast, as claimed. ■

2. (Rigid Robot Control). Consider a frictionless rigid 2-link robot manipulator (or double pendulum) with control torques $u_1$ and $u_2$ applied at the joints. The dynamics of such a robot-arm may be obtained via the Euler-Lagrange formalism which yields:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + k(\theta) = u,$$

where $\theta = (\theta_1, \theta_2)^T$, $\theta_i$ are the joint angles, and $u = (u_1, u_2)^T$. The term $k(\theta)$ represents the gravitational force and $C(\theta, \dot{\theta})$ reflects the centripetal and Coriolis forces. The matrix $M(\theta)$ has everywhere positive determinant.

(a) Using as outputs the angles $\theta$, find its relative degree and convert it to a normal form.

(b) What are the zero dynamics of the system?

Solution.

(a) Let

$$u = C(\theta, \dot{\theta}) + k(\theta) + M(\theta)v,$$

where $v \in \mathbb{R}^2$ is the new input. Substituting this into the system equation yields $M(\theta)\ddot{\theta} = M(\theta)v$, or, since $\det M(\theta) \neq 0$,

$$\ddot{\theta} = v,$$

which corresponds to the linear controllable dynamics

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v,$$

where $0$ is the $2 \times 2$ zero matrix and $I$ is the $2 \times 2$ identity matrix.

Therefore, the system is linearizable by state feedback only. It is not hard to see that the vector relative degree of the new, and therefore old, system is $(2,2)$.

Note that here $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2) \in S^1 \times S^1$, where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle, and $\dot{\theta} \in \mathbb{R}^2$. Therefore, the dynamics are not globally linear since the state space, $S^1 \times S^1 \times \mathbb{R}^2$ is not a linear space.

(b) It follows from (a) that the zero dynamics are trivial.

3. (Invariant Distributions, Continued). Recall that the notion of a distribution invariant with respect to a vector field was defined in Homework 5.
(a) Let $X_1, \ldots, X_k$ be analytic vector fields on $\mathbb{R}^n$. Define $\mathcal{L}$ to be the vector space of all vector fields of the form $\sum_{i=1}^l \alpha_i(x) Y_i(x)$, where $\alpha_i$ are smooth functions and $Y_i$ are vector fields of the form

$$[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{r-1}}, X_{i_r}] \cdots]],$$

i.e., nested Lie brackets of vector fields $X_r$ ($r \geq 1$, $1 \leq i_1, \ldots, i_r \leq k$). For each $x \in \mathbb{R}^n$, define a distribution $\mathcal{C}$ by

$$\mathcal{C}(x) = \{ Z(x) : Z \in \mathcal{L} \}.$$ 

Show that $\mathcal{C}$ is invariant with respect to $X_1, \ldots, X_k$. Moreover, show that if $D$ is any other distribution containing $X_1, \ldots, X_k$ and invariant with respect to $X_1, \ldots, X_k$, then $\mathcal{C}(x) \subseteq D(x)$, for all $x \in \mathbb{R}^n$. That is, $\mathcal{C}$ is the smallest distribution with these two properties. We write $\mathcal{C} = \text{Lie}(X_1, \ldots, X_k)$ to indicate that $\mathcal{C}$ is the “Lie algebra” generated by vector fields $X_1, \ldots, X_k$.

(b) Show that $\mathcal{C}$ is involutive.

(c) Consider an affine control system $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f$ and $g = [g_1] \cdots [g_m]$ are analytic. The accessibility distribution of this system is equal to $\text{Lie}(f, g_1, \ldots, g_m)$. Now consider the system

$$\begin{align*}
\dot{x}_1 &= x_2^2 \\
\dot{x}_2 &= u,
\end{align*}$$

find its accessibility distribution and the set of reachable points.

Solution.

(a) Recall from Homework 4 that $\mathcal{C}$ is invariant with respect to a vector field $X$ iff for every vector field $Y$ in $\mathcal{C}$, $[X, Y] \in \mathcal{C}$. Therefore, by construction of $\mathcal{C}$, it follows that it is invariant with respect to $X_1, \ldots, X_k$.

To show that $\mathcal{C}$ is the smallest distribution which contains $X_1, \ldots, X_k$ and is invariant with respect to them, assume that $D$ in any distribution containing $X_1, \ldots, X_k$ and invariant relative to them. Then, by invariance,

$$[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{r-1}}, X_{i_r}] \cdots]] \in D,$$

for all $r \geq 1$ and $1 \leq i_j \leq k$. Therefore $\mathcal{C} \subseteq D$, i.e., $\mathcal{C}$ is smaller than $D$.

(b) To show that $\mathcal{C} = \text{Lie}(X_1, \ldots, X_k)$ is involutive it suffices to show that the Lie bracket of any two vector fields of the form $[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{r-1}}, X_{i_r}] \cdots]]$, for any $r \geq 1$, is in $\mathcal{C}$. So let

$$Y = [Y_1, [Y_2, \cdots, [Y_{r-1}, Y_r] \cdots]] \text{ and } Z = [Z_1, [Z_2, \cdots, [Z_{s-1}, Z_s] \cdots]],$$

where $Y_i, Z_j \in \{X_1, \ldots, X_k\}$, and $r, s \geq 1$. We will show that $[Y, Z] \in \mathcal{C}$. This will be done by induction on $r$, i.e. the length of $Y$.

Clearly, the statement is true if $r = 1$, for any $s$. Suppose it is true for $r = l - 1$ and all $s$. Assuming $r = l$, it then follows from the Jacobi identity that

$$[Y, Z] = -[\tilde{Y}, [Y_1, Z]] + [Y_1, [\tilde{Y}, Z]],$$

where $\tilde{Y} = [Y_2, \cdots, [Y_{l-1}, Y_l] \cdots]$. Since the length of $\tilde{Y}$ is $l - 1$, it follows by the induction hypothesis that the right hand side is in $\mathcal{C}$, which completes the proof.
(c) The drift and control vector fields of the given system are
\[ f(x) = x_2^2 e_1, \quad g(x) = 2e_2, \]
where \( e_1 = (1, 0, \ldots, 0)^T, e_2 = (0, 1, 0, \ldots, 0)^T, \) etc.

Since
\[ [f, g] = -2x_2e_1, \quad [[f, g], g] = 2e_1, \]
and \( g \) and \( [[f, g], g] \) span the tangent space of \( \mathbb{R}^2 \) at any point, we have \( \dim C(x) = 2 \) everywhere. Therefore, the system is locally accessible. However, since \( \dot{x}_1 = x_2^2 \geq 0 \), the first coordinate of every trajectory is non-decreasing, and the reachable sets are
\[ \mathcal{R}(x_1, x_2) = [x_1, \infty) \times \mathbb{R}, \]
so the system is not controllable. So in this example, full-rank accessibility distribution + bad drift = non-controllability.

4. (Rocket Outside the Atmosphere). Consider the dynamics of a rocket outside the atmosphere. The forces which act on the rocket are the gravitational force and the force as delivered by the rocket motor. The control variable is the angle \( \alpha \) expressing the direction of the force as delivered by the rocket motor. Take state space variables \( x_1 = r, x_2 = \theta, x_3 = \dot{r}, x_4 = \dot{\theta} \), where \( (r, \theta) \) are polar coordinates in the plane containing the center of the earth and the trajectory of the rocket. Then the dynamics are given by
\[ \begin{align*}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{gR^2}{x_1^2} + \frac{T}{m} \cos u + x_1 x_4 \\
\dot{x}_4 &= -\frac{2x_3 x_4}{x_1} + \frac{T}{mx_1} \sin u.
\end{align*} \]
Here \( m \) is the mass of the rocket, \( g \) the gravitational constant, and \( R \) the radius of the earth. Note that the system is not affine.

(a) To obtain an affine system, extend the given system by adding the equation \( \dot{u} = w \) and taking \( z = (x, u) \) to be the new state space variable, and \( w \) to be the new control variable. Write down the extended system (E).

(b) Let \( f \) and \( g \) be the drift and input vector field of the extended system respectively. Compute \([f, g], [f, [f, g]], [f, [f, [f, g]]], \) and \([g, [f, g]]\).

(c) Show that (E) is not exactly feedback linearizable.

Proof.

(a) The extended system is
\[ \dot{z} = f(z) + g(z)w, \]
where \( z = (x, u) \). A calculation yields

\[
\begin{bmatrix}
  x_3 \\
  x_4 \\
  -\frac{gR^2}{x_1^2} + x_1 x_4^2 + \frac{T}{m} \cos u \\
  -\frac{2x_3 x_4}{x_1} + \frac{T}{m x_1} \sin u \\
  0
\end{bmatrix}, \quad g(z) = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix},
\]

\[
[f, g](z) = \begin{bmatrix}
  0 \\
  0 \\
  \frac{T}{m} \sin u \\
  -\frac{2x_3 x_1}{m} \cos u \\
  0
\end{bmatrix}, \quad [f, [f, g]](z) = \begin{bmatrix}
  \frac{T}{m} \sin u \\
  \frac{2R^2 x_3 x_1}{x_1^2} \sin u + \frac{x_4}{x_1} \cos u \\
  \frac{2x_3 x_1}{m} \cos u \\
  0 \\
  0
\end{bmatrix},
\]

and

\[
[f, [f, [f, g]]](z) = \begin{bmatrix}
  -\cos u \\
  -\frac{2x_4}{x_1} \sin u \\
  \frac{2R^2 x_3 x_1}{x_1^2} \sin u + \frac{x_4}{x_1} \cos u \\
  \frac{2x_3 x_1}{m} \cos u + \frac{T}{m x_1^2} (2 \sin^2 u - 1)
\end{bmatrix}.
\]

It follows that the distribution \( G = \text{span}\{g, [f, g], [f, [f, g]], [f, [f, [f, g]]]\} \) is not involutive since (by some linear algebra)

\[
[g, [f, g]](z) = \begin{bmatrix}
  0 \\
  0 \\
  \frac{T}{m} \cos u \\
  \frac{T}{m x_1} \sin u \\
  0
\end{bmatrix} \notin G(z).
\]

Therefore, by the theorem on exact (or full) feedback linearizability for SISO systems, it follows that (2) is not exactly feedback linearizable.

5. (Connecting Points). Consider the control system (S) on \( \mathbb{R}^3 \)

\[
\begin{align*}
  \dot{x} &= u \\
  \dot{y} &= v \\
  \dot{z} &= -vx,
\end{align*}
\]

where \( u, v \in \mathbb{R} \) are the inputs.

(a) Show that (S) is controllable. Show that any two points can be joined by a piecewise smooth control trajectory, corresponding to a piecewise input.

(b) Given any two points \( p_0, p_1 \in \mathbb{R}^3 \), construct a smooth control trajectory connecting \( p_0 \) and \( p_1 \).

(c, optional) For \( p, q \in \mathbb{R}^3 \), define \( d_*(p, q) \) as the infimum of the length of all control trajectories connecting \( p \) and \( q \). Show that \( d_* \) defines a metric (i.e., distance) on \( \mathbb{R}^3 \), that is, show that \( d_*(p, q) \geq 0 \), \( d_*(q, p) = d_*(p, q) \), and \( d_*(p_0, p_2) \leq d_*(p_0, p_1) + d_*(p_1, p_2) \). \( d_* \) is called a sub-Riemannian (or Carnot-Carathéodory) metric.
Then show that (at least locally speaking) there exists a constant \( C > 0 \) such that for any two points \( p, q \) with different \( z \)-coordinates,

\[
d_*(p, q) \leq C|p - q|^{1/2},
\]

where \(| \cdot |\) denotes the Euclidean norm.

**Solution.**

(a) The control vector fields of the system are

\[
X = e_1 \quad \text{and} \quad Y = e_2 - xe_3.
\]

Their Lie bracket is

\[
Z = [X, Y] = -e_3.
\]

Therefore, the accessibility distribution of the system, \( C(p) = \text{span}\{X(p), Y(p), Z(p)\} \) is of dimension three at every point \( p \in \mathbb{R}^3 \). Since \( \mathbb{R}^3 \) is connected and the inputs are unconstrained, by Chow’s (or Chow-Rashevski, or controllability) theorem, the system is controllable.

In finding a piecewise smooth control trajectory connecting some \( p_0 \) to some \( p_1 \), the most challenging case is when \( p_1 - p_0 \) is parallel to \( Z \). (The other two directions can be accessed more or less directly via the flows of \( X \) and \( Y \).) It is not hard to check that for every \( p \in \mathbb{R}^3 \),

\[
(Y^{-t} \circ X^{-t} \circ Y^t \circ X^t)(p) = Z^{t^2}(p) = p + (0, 0, -t^2).
\]

Therefore, to drive the system from \( p_0 \) to \( p_0 + (0, 0, -t^2) \), the sequence of inputs should be \((1, 0), (0, 1), (-1, 0), (0, -1)\), each one applied for time \( t \).

(b) Observe first that a smooth curve \((x(t), y(t), z(t))\) is a trajectory of the given control system if and only if

\[
\dot{z} + xy = 0.
\]

Let \( p_0 = (x_0, y_0, z_0) \) and \( p_1 = (x_1, y_1, z_1) \) be arbitrary points in \( \mathbb{R}^3 \). First we will construct a smooth trajectory \( c(t) = (x(t), y(t)) \), \( 0 \leq t \leq 1 \), of the system \( \dot{x} = u, \dot{y} = v \) connecting \((x_0, y_0)\) and \((x_1, y_1)\) – i.e., \( c(0) = (x_0, y_0) \) and \( c(1) = (x_1, y_1) \) – such that

\[
\int_0^1 x(t)\dot{y}(t) \, dt = z_0 - z_1.
\]

We seek such a trajectory among curves of constant curvature, that is, in the form

\[
x(t) = a_0 + a_1 t + a_2 t^2, \quad y(t) = b_0 + b_1 t + b_2 t^2.
\]

Clearly, \( a_0 = x_0 \) and \( b_0 = y_0 \). Furthermore, \( c(1) = (x_1, y_1) \) and (4) give us a system of three equations with four unknowns \( a_1, a_2, b_1, b_2 \),

\[
\begin{align*}
a_1 + a_2 &= x_1 - x_0, \\
b_1 + b_2 &= y_1 - y_0, \\
x_0b_1 + x_0 + \frac{1}{2}a_1b_1 + \frac{1}{3}(a_2b_1 + 2a_1b_2) &= z_0 - z_1.
\end{align*}
\]
Assuming \( x_0 \neq x_1 \) (otherwise proceed analogously), we can choose \( a_1 = 0 \) which implies \( a_2 = x_1 - x_0 \). This leaves us with two linear equations for \( b_1, b_2 \); their solutions are

\[
 b_1 = 3 \frac{z_1 - z_0 - x_0(y_1 - y_0)}{x_1 - x_0}, \quad b_2 = y_1 - y_0 - b_1.
\]

This completes the construction of \( c(t) \).

Now, to construct a smooth control trajectory \( p(t), 0 \leq t \leq 1 \), of the original system connecting \( p_0 \) and \( p_1 \), choose \( x(t) \) and \( y(t) \) for the first two coordinates of \( p(t) \), and let the third coordinate be

\[
 z(t) = z_0 - \int_0^t x(s)\dot{y}(s) \, ds.
\]

Then \( z(1) = z_0 - (z_0 - z_1) = z_1 \), by (4), and \( \dot{z}(t) + x(t)\dot{y}(t) = 0 \), for all \( 0 \leq t \leq 1 \). Therefore, by the remark from the beginning, \( p(t) \) is a trajectory of the system, it is smooth, and it connects \( p_0 \) and \( p_1 \). This completes the solution.

Observe that the corresponding inputs are

\[
 u(t) = \dot{x}(t) = a_1 + 2a_2t, \quad v(t) = \dot{y}(t) = b_1 + 2b_2t.
\]

(c) It follows straight from the definition that \( d_*(p, q) = 0 \) iff \( p = q \), and that \( d_*(q, p) = d_*(p, q) \). To show the triangle inequality, let \( c_i \) be a control trajectory connecting \( p_i \) to \( p_{i+1} \), for \( i = 0, 1 \). Then \( c = c_0 + c_1 \) is a control trajectory connecting \( p_0 \) to \( p_2 \). Therefore, \( d_*(p_0, p_2) \leq L(c) = L(c_0) + L(c_1) \). Taking the infimum of the right hand side over all \( c_0 \) and \( c_1 \) yields \( d_*(p_0, p_2) \leq d_*(p_0, p_1) + d_*(p_1, p_2) \).

Now let \( q = p + (0, 0, -t^2) \), for some \( t \). Then \( |p - q| = t^2 \). By (3), \( d_*(p, q) \) ≤ the length of the usual zig-zag control trajectory \( c \) consisting of two arcs tangent to \( \pm X \) and two arcs tangent to \( \pm Y \), each of length \( \leq M|t| \), where \( M \) is the maximum of the norms of \( X \) and \( Y \) on some small neighborhood of \( p \) containing \( c \). Therefore,

\[
 d_*(p, q) \leq 4M|t| = 4M|p-q|^{1/2},
\]

as claimed.

Observe that if \( q = X^t(p) \) or \( q = Y^t(p) \), then \( d_*(p, q) \leq M|t| \leq C|p - q| \), for some constant \( C \), since \( |p - q| \) is approximately \( |t| \).