Homework #4


a) Consider $\dot{x} = f(x)$. Prove that $x = 0$ is globally asymptotically stable if there exist $P, Q > 0$ such that

$$P \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} P = -Q$$

(4)

Proof: The key for this proof is that (4) holds for any $x \in \mathbb{R}^n$.

Before starting the proof, I first state the "mean value" theorem in $\mathbb{R}^n$ (for multiple value functions).

Theorem. Let $U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. $U$ starlike with respect to $y$, $f \in C^1(U)$. Then, we have

$$f(x) - f(y) = \nabla f(y) (x - y)$$

where $\nabla f = \sum_t Df(x_0 + ty + tx) dt$.

The proof is simple.

$$\int (x_1 - x_2) = \int \frac{d}{dt} f_t(x_1 + ty + tx) dt$$

(chain rule)

$$= \int \nabla f(y) (x - y) dt$$

$$= \nabla f(y) (x - y) \quad \square$$

Now, let's come back to the initial problem. For the system $\dot{x} = f(x)$, choose Lyapunov function $x^T P x = V(x)$.

Then $V(x) = \frac{1}{2} x^T P x + x^T P f(x)$.

Now, use mean value theorem for $f(x)$ by letting $y = 0$.

$$V(x) = \left( \int \frac{d}{dt} f_t(x) dt \right) \cdot x \quad \text{ (Here, } \frac{d}{dt} f_t(x) = \frac{Df}{dx}(t(x)) \right)$$

Plug this in $V(x)$, we get

$$V(x) = x^T \left( \int \frac{d}{dt} f_t(x) dt + \frac{d}{dx} (f(x) P) \right) x$$

$$= x^T \left( \int \frac{d}{dt} f_t(x) dt + \frac{d}{dx} (f(x) P) \right) x$$

$$= -x^T Q x$$

Then, we get $V(x)$ is p.d.f. and $-V(x)$ is also p.d.f.

So, by Lyapunov Basic theorem, $x = 0$ is globally asymptotically stable.
b) Write \( f(x) = A(x)x \) for some \( n \times n \) matrix \( A(x) \). Show that \( x = 0 \) is globally asymptotically stable if there exist \( P, Q > 0 \) such that

\[
P A(x) + A(x)^T P = -Q.
\]

Proof: This proof is quite direct than that of (A).

Choose Lyapunov function \( V(x) = x^T P x \).

\[
\dot{V}(x) = x^T P f(x) + f(x)^T P x
\]

\[
= x^T (PA(x) + A(x)^T P) x
\]

\[
= -x^T Q x
\]

Then \( V(x) \) is p.d.f and \( -\dot{V}(x) \) is p.d.f. So, by Lyapunov basic theorem, \( x = 0 \) is globally asymptotically stable.
2. The nonlinear second order system:
\[
\begin{align*}
    x_1 &= -x_1 + \alpha x_1 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \\
    x_2 &= x_1 + \alpha x_2 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2)
\end{align*}
\]

i) Show that the linearization (and hence the indirect method of Lyapunov) is inconclusive to determine the stability of the origin.

**proof:** The linearization of this system around the origin is:
\[
\begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix} = \begin{pmatrix}
    0 & -1 \\
    1 & 0
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

The eigenvalues of the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) is \( \pm j \), just on the jω axis. Thus, the linearization system is not sufficient to determine the stability.

ii) Use the direct Lyapunov method to study the stability of the origin between \( \alpha = 1 \) and \( \alpha = -1 \).

One solution:
Choose \( V(x) = x_1^2 + x_2^2 \). (P.d.f.)

Then, by direct calculation,
\[
\dot{V}(x) = 2\alpha x_1^2 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) - 2x_1 x_2
\]
\[
+ 2\alpha x_2^2 (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) + 2x_1 x_2
\]
\[
= 2\alpha (x_1^2 + x_2^2) \sin(x_1^2 + x_2^2)
\]

When \( |x_1|^2 = x_1^2 + x_2^2 \) is small enough, we have:
\[
\sin(x_1^2 + x_2^2) \lesssim \sin(x_1^2 + x_2^2) \leq x_1^2 + x_2^2
\]

Then, when \(-1 < \alpha < 0\), \( \dot{V}(x) \leq + \alpha (x_1^2 + x_2^2)^2 \), i.e., \( \dot{V}(x) \)

is l.p.d.f. Therefore, \( x=0 \) is uniformly asymptotically stable.

when \( 0 < \alpha < 1 \), \( \dot{V}(x) \geq \alpha (x_1^2 + x_2^2)^2 \), \( \dot{V}(x) \) is

l.p.d.f. Then, by the instability theorem, origin \( x=0 \)

is unstable.

when \( \alpha = 0 \). The system is just the linearization system

\( x=0 \) is stable (critically stable).
3) Rayleigh's equation:
\[ \begin{align*}
\dot{x}_1 &= -\epsilon (x_1^2/2 - x_1) + x_2 \\
\dot{x}_2 &= -x_1
\end{align*} \]

i). Show that the limit cycle does not lie completely inside the strip \(-1 \leq x_1 \leq 1\).

Proof: Let's prove this by Bendixson's theorem.
\[ \text{div}(\mathbf{f}) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \epsilon (1 - x_1^2) \]
When \(-1 \leq x_1 \leq 1\), \(\text{div}(\mathbf{f}) \geq 0\) (with "holding only on \(x_1^2 = 1\)).
Then, by Bendixson's theorem, the strip \(-1 \leq x_1 \leq 1\) does not contain any closed orbit of the system. So the limit cycle cannot lie completely inside \(-1 \leq x_1 \leq 1\).

ii). Using a suitable Lyapunov function, say, \(x_1^2 + x_2^2\), show that it lies outside a circle of radius \(\sqrt{3}\).

Proof:
\[ V(x) = x_1^2 + x_2^2 \]
Through direct calculation,
\[ \dot{V}(x) = \frac{\epsilon}{2} \sum x_i^2 (3 - x_i^2) \]
Now let's prove by contradiction that there is no closed orbit lying completely inside the strip \(-1 \leq x_1 \leq 1\) (This strip includes \(x_1^2 + x_2^2 \leq 3\)!)
Suppose there is such a closed orbit (not trivial) \(x(t), 0 \leq t \leq T\), lying completely inside \(x_1^2 \leq 3\). Certainly, we have \(V(x(0)) = V(x(T))\).
while, we also have
\[ V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t)) \, dt \]
\[ = V(x(0)) + \int_0^T \frac{\epsilon}{2} \sum x_i^2 (3 - x_i^2) \, dt \]
Then we have \(\int_0^T \frac{\epsilon}{2} (3 - x_i^2) \, dt = 0\)
Since, when $x^2 \leq 3$, $x^2 (3 - x^2) \geq 0$, from the last equation we get $x^2 (3 - x^2) = 0$. This means

$x(t) = 0$, $+\sqrt{3}$ or $-\sqrt{3}$. This is a contradiction to the assumption that $x(t)$ is an nontrivial closed orbit. Therefore, no nontrivial closed orbit lies completely inside strip $x^2 \leq 3$, except $x = 0$.\[\]
4) Consider \( x = f(x, t) \), let \( V(t, x) \) be a Lyapunov function satisfying 
\[
\dot{V}(t, x) \leq \nabla V(x) \cdot \nabla V(x) - \lambda V(x, l) \leq 0.
\]
Further, assume that \( \exists \beta > 0 \) s.t. 
\[
\int_{t_0}^{t_\delta} V(t, x(t)) dt \leq \frac{1}{\lambda} \beta \left( \frac{t - t_0}{\delta} \right)^p,
\]
for all \( t > t_0 \).
Show that \( \dot{V}(t) \) converges exponentially to 0.
Proof: 
\[
\begin{align*}
\int_{t_0}^{t+\delta} \dot{V}(t, x) dt & \leq -\lambda \int_{t_0}^{t+\delta} V(t, x(t)) dt,
\end{align*}
\]

\[
\Rightarrow V(t_0 + \delta, x(t_0 + \delta)) - V(t_0, x(t_0)) \leq -\lambda \int_{t_0}^{t+\delta} V(t, x(t)) dt.
\]

\[
\Rightarrow V(t, x(t)) \leq (1 - \frac{\lambda \delta}{\lambda}) V(t, x(t_0)) \quad \forall t \geq t_0.
\]

Note, since \( V(t, x(t_0)) \to 0 \) for all \( t \), we have 
\[
(1 - \frac{\lambda \delta}{\lambda}) < 1. \quad (\lambda, \delta > 0).
\]

Since \( V(t, x(t)) \leq 0 \), \( V(t, x(t_0)) \leq V(t_0, x(t_0)) \) \( \forall t \geq t_0 \).
Then when \( t_0 + k \delta \leq t \leq t_0 + (k+1) \delta, \quad k \geq 1 \),

(1) \( V(t, x(t)) \leq (1 - \frac{\lambda \delta}{\lambda})^k V(t - k \delta, x(t - k \delta)) \)

\[
\leq (1 - \frac{\lambda \delta}{\lambda})^k V(t_0, x(t_0)).
\]

\[
\leq (1 - \frac{\lambda \delta}{\lambda}) \left( \frac{t - t_0}{\delta} \right)^p.
\]

(2) \( V(t_0, x(t_0)) \to 0 \) for \( t_0 \leq t \leq t_0 + \delta \).

How to combine (1), (2) to only one inequality? (The reason that we cannot simply use (1) for all \( t \geq 0 \) is that 
\[
1 - \frac{\lambda \delta}{\lambda} < 1.
\]

While, if we suppose \( f(x, t) \) is Lipschitz continuous, this trouble disappears, since \( V(t, x(t)) \)
can not reach \( x = 0 \) in finite time, i.e. necessarily, 
\[
1 - \frac{\lambda \delta}{\lambda} \text{ must be greater than zero}.
\]

Since \( 1 - \frac{\lambda \delta}{\lambda} < 1 \), \( \exists \beta > 0 \) s.t. \( 0 < 1 - \frac{\lambda \delta}{\lambda} + \beta < 1 \).
Denote \( 1 - \frac{\lambda \delta}{\lambda} + \beta \) by \( \beta' \). Now we could put (1), (2) into one inequality:

\[
V(t_0, x(t_0)) \leq \beta \left( \frac{t - t_0}{\delta} \right)^p \cdot V(t_0, x(t_0))
\]

\[
\Rightarrow \lambda \beta \left( \frac{t - t_0}{\delta} \right)^p \leq \beta \left( \frac{t - t_0}{\delta} \right)^p \cdot \lambda \beta \left( \frac{t - t_0}{\delta} \right)^p
\]
\[ |x(t)|^2 \leq \frac{1}{\beta x_1} \cdot e^{\left( \frac{1}{\beta} \ln \beta \right) (t-t_0)} |x(t_0)|^2 \]
\[ |x(t)| \leq \sqrt{\frac{1}{\beta x_1}} \cdot e^{\left( \frac{1}{\beta} \ln \beta \right) (t-t_0)} |x(t_0)| \]

Since \( 0 < \beta < 1 \), \( \frac{1}{\beta} \ln \beta < 0 \).

Let \( \lambda = -\frac{1}{2} \ln \beta > 0 \), \( m = \sqrt{\frac{1}{\beta x_1}} \), we have

\[ |x(t)| \leq m \cdot e^{-\lambda (t-t_0)} |x(t_0)| \quad \text{if} \quad t \geq t_0. \]

Therefore, \( x = 0 \) is globally exponentially stable.