Homework is due at the beginning of class on the due day!

Problems:

1. **(Smooth Stabilizability).** Assume that the control system
\[ \dot{x} = f(x, u) \]
is smoothly exponentially stabilizable at \( 0 \in \mathbb{R}^n \), that is, there exists a smooth control law \( u = k(x) \) such that \( 0 \) is a locally exponentially stable equilibrium of the closed loop system \( \dot{x} = f(x, k(x)) \). Prove that the extended system
\[ \begin{align*}
\dot{x} &= f(x, z) \\
\dot{z} &= h(x, z) + u,
\end{align*} \]
where \( h \) is a smooth function, is also exponentially stabilizable by smooth feedback. (**Hint:** Try a Lyapunov function \( W(x, z) = V(x) + \frac{1}{2} |z - k(x)|^2 \), where \( V \) is an appropriate Lyapunov function for the original system.)

2. **(Rigid Robot Control).** Consider a frictionless rigid 2-link robot manipulator (or double pendulum) with control torques \( u_1 \) and \( u_2 \) applied at the joints. The dynamics of such a robot-arm may be obtained via the Euler-Lagrange formalism which yields:
\[ M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + k(\theta) = u, \]
where \( \theta = (\theta_1, \theta_2)^T \), \( \theta_i \) are the joint angles, and \( u = (u_1, u_2)^T \). The term \( k(\theta) \) represents the gravitational force and \( C(\theta, \dot{\theta}) \) reflects the centripetal and Coriolis forces. The matrix \( M(\theta) \) has everywhere positive determinant.

(a) Using as outputs the angles \( \theta \), find its relative degree and convert it to a normal form.
(b) What are the zero dynamics of the system?

3. **(Invariant Distributions, Continued).** Recall that the notion of a distribution invariant with respect to a vector field was defined in Homework 5 (problem 5).

(a) Let \( X_1, \ldots, X_k \) be analytic vector fields on \( \mathbb{R}^n \). Define \( \mathcal{L} \) to be the vector space of all vector fields of the form \( \sum_{i=1}^t \alpha_i(x)Y_i(x) \), where \( \alpha_i \) are smooth functions and \( Y_i \) are vector fields of the form
\[ [X_{i_1}, [X_{i_2}, \cdots, [X_{i_{r-1}}, X_{i_r}], \cdots]], \]
i.e., nested Lie brackets of vector fields \( X_i \) (\( r \geq 1, 1 \leq i_1, \ldots, i_r \leq k \)). For each \( x \in \mathbb{R}^n \), define a distribution \( \mathcal{C} \) by
\[ \mathcal{C}(x) = \{ Z(x) : Z \in \mathcal{L} \}. \]
Show that $C$ is invariant with respect to $X_1, \ldots, X_k$. Moreover, show that if $D$ is any other distribution containing $X_1, \ldots, X_k$ and invariant with respect to $X_1, \ldots, X_k$, then $C(x) \subseteq D(x)$, for all $x \in \mathbb{R}^n$. That is, $C$ is the smallest distribution with these two properties. We write $C = \text{Lie}(X_1, \ldots, X_k)$ to indicate that $C$ is the “Lie algebra” generated by vector fields $X_1, \ldots, X_k$.

(b) Show that $C$ is involutive.

(c) Consider an affine control system $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f$ and $g = [g_1 \cdots g_m]$ are analytic. The accessibility distribution of this system is equal to $\text{Lie}(f, g_1, \ldots, g_m)$. Now consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2^2 \\
\dot{x}_2 &= u,
\end{align*}
\]

find its accessibility distribution and the set of reachable points.

4. (Rocket Outside the Atmosphere). Consider the dynamics of a rocket outside the atmosphere. The forces which act on the rocket are the gravitational force and the force as delivered by the rocket motor. The control variable is the angle $\alpha$ expressing the direction of the force as delivered by the rocket motor. Take state space variables $x_1 = r$, $x_2 = \theta$, $x_3 = \dot{r}$, $x_4 = \dot{\theta}$, where $(r, \theta)$ are polar coordinates in the plane containing the center of the earth and the trajectory of the rocket. Then the dynamics are given by

\[
\begin{align*}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{gR^2}{x_1^2} + \frac{T}{m} \cos u + x_1 x_4 \\
\dot{x}_4 &= -\frac{2x_3 x_4}{x_1} + \frac{T}{m x_1} \sin u.
\end{align*}
\]

Here $m$ is the mass of the rocket, $g$ the gravitational constant, and $R$ the radius of the earth. Note that the system is not affine.

(a) To obtain an affine system, extend the given system by adding the equation $\dot{u} = w$ and taking $z = (x, u)$ to be the new state space variable, and $w$ to be the new control variable. Write down the extended system (E).

(b) Let $f$ and $g$ be the drift and input vector field of the extended system respectively. Compute $[f, g], [f, [f, g]], [f, [f, [f, g]]], \text{ and } [g, [f, g]]$.

(c) Show that (E) is not exactly feedback linearizable.

5. (Connecting Points). Consider the control system (S) on $\mathbb{R}^3$

\[
\begin{align*}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= -vx,
\end{align*}
\]

where $u, v \in \mathbb{R}$ are the inputs.

(a) Show that (S) is controllable. Show that any two points can be joined by a piecewise smooth control trajectory, corresponding to a piecewise input.
(b) Given any two points $p_0, p_1 \in \mathbb{R}^3$, construct a smooth control trajectory connecting $p_0$ and $p_1$.

(c, optional) For $p, q \in \mathbb{R}^3$, define $d_*(p, q)$ as the infimum of the length of all control trajectories connecting $p$ and $q$. Show that $d_*$ defines a metric (i.e., distance) on $\mathbb{R}^3$, that is, show that $d_*(p, q) \geq 0$, $d_*(p, q) = 0$ iff $p = q$, $d_*(q, p) = d_*(p, q)$, and $d_*(p_0, p_2) \leq d_*(p_0, p_1) + d_*(p_1, p_2)$. $d_*$ is called a sub-Riemannian (or Carnot-Carathéodory) metric. Then show that (at least locally speaking) there exists a constant $C > 0$ such that for any two points $p, q$ with different $z$-coordinates,

$$d_*(p, q) \leq C|p - q|^{1/2},$$

where $|\cdot|$ denotes the Euclidean norm.