1. Tracking
Let \( e = y - r \). Then \( \dot{e} = \dot{y} - \dot{r} = \dot{x} - \dot{r} = x + u - \dot{r} \).

Want \( \dot{e} = -\lambda e \), \( \lambda > 0 \) \( \Rightarrow \) \( x + u - \dot{r} = -\lambda e \) \( \Rightarrow \) \( u = -(1 + \lambda)y + \lambda r + \dot{r} \) for some \( \lambda > 0 \).

Thus the control is \( u = -(1 + \lambda)y + \lambda r + \dot{r} \) for some \( \lambda > 0 \).

Simulation should show that \( y(t) \) approaches \( r(t) \) asymptotically. If you plot \( e(t) \), it will decay exponentially.

2. Infinite horizon optimal control for nonlinear systems

Part a) and b) derivations closely follow the derivation for finite-time horizon as in the course note Section 10.2.

Let the value function be
\[
J^0(x) := \min_u \int_t^\infty \ell(x(s), u(s)) ds, \quad x(t) = x.
\]

where \( x \) is the initial state at time \( t \) (little abuse of notation \( x \) here but that’s what does in course note and other textbooks; to understand, you can call it \( z \) and let \( z \) have the dynamic of \( x \): \( \dot{z} = f(z, u) \)).

Because the integral is infinite horizon (integration up to infinity), the value function does not depend on the initial time \( t \) but is a function of the initial state \( x \) only.

Principal of optimality:
\[
J^0(x) = \min_u \left\{ \ell(x, u(t)) \Delta t + J^0(x(t_m)) + \frac{dJ^0}{dx} \Delta x + H.O.T \right\}
\]

Let \( t_m = t + \Delta t, \ x(t_m) = x(t) + \Delta x = x + \Delta x \), we have
\[
J^0(x) = \min_u \left\{ \ell(x, u(t)) \Delta t + J^0(x(t_m)) + \frac{dJ^0}{dx} \Delta x + H.O.T \right\}
\]

where \( H.O.T \) are higher order terms of \( \Delta t, \Delta x \) such that \( \lim(H.O.T)/\Delta t \to 0 \) as \( \Delta t \to 0 \).

Cancelling \( J^0(x) \) and dividing both sides of (5) by \( \Delta t \) and letting \( \Delta t \to 0 \), we have
\[
0 = \min_u \left\{ \ell(x, u) + \frac{dJ^0}{dx} f(x, u) \right\} \quad \text{(HJB equation)}
\]

by the fact that \( \lim_{\Delta t \to 0} \Delta x/\Delta t = f(x, u) \).

Apply the above result for the system \( \dot{x} = u \) with cost function \( V(u) = \int_0^\infty x^4 + u^2 dt \), we have
\[
0 = \min_u \{x^4 + u^2 + \frac{dJ^0}{dx} u\}
\]

The right-hand side is quadratic in \( u \) so it is minimized when
\[
u^* = -\frac{1}{2} \frac{dJ^0}{dx}.
\]
Now, the value of the right-hand side with this \( u^* \) must be 0 so

\[
0 = x^4 - \frac{1}{4} \left( \frac{dJ^o}{dx} \right)^2
\]

that gives \( \frac{dJ^o}{dx} = 2x^2 \) or \(-2x^2\) and hence, \( J^o(x) = 2x^3/3+C \) or \(-2x^3/3+C\), respectively. Since \( J^o(x) \geq 0 \) (from the definition of \( J^o \)) and \( J^o(x) = 0 \) when \( x = 0 \) (since then \( u^* = 0 \Rightarrow J^o = 0 \)), we conclude that we must have \( J^o(x) = 2x^2|x|/3 \).

The optimal control is

\[
u^* = \begin{cases} 
-x^2 & \text{if } x \geq 0 \\
x^2 & \text{if } x < 0
\end{cases}
\]

or in short, \( u = -x|x| \).

Comment: The HJB equation is the centerpiece of the section on optimal state feedback control. It is important to know the derivation. This problem helps by going through the derivation once more (you should be able to do the derivation without looking at the notes). Also, the theorem says that the optimal solution \( J^o \) satisfies the HJB equation (a necessary condition only, not sufficient) so we needs to use further reasoning to get the actual optimal solution from the HJB equation.

3. Scalar linear systems

The infinite horizon optimal problem of linear systems: minimize \( \int_0^\infty x^T Qx + u^T Ru \) dt subject to \( \dot{x} = Ax + Bu \), \( x(0) = x_0 \).

The optimal solution is \( u = -R^{-1}B^TPx \) where \( P \) is a solution in the class of positive semi-definite matrices of the ARE equation

\[
A^T P + PA + Q - PBR^{-1}B^TP = 0.
\]

For the scalar system and the cost function in the problem, \( A = a, B = 1, Q = 1, R = r \) so the ARE equation is

\[
2ap + 1 - p^2/r = 0,
\]

which gives the solution \( p = ar \pm r\sqrt{a^2 + 1/r} \). We only takes \( p \geq 0 \), so \( p^* = ar + r\sqrt{a^2 + 1/r} \). The optimal control is \( u^* = -(a + \sqrt{a^2 + 1/r})x \) and the closed-loop system is \( \dot{x} = -\sqrt{a^2 + 1/r} x \). Closed-loop eigenvalue is \( \lambda = -\sqrt{a^2 + 1/r} \).

(a) \( r \to 0 \), then \( \lambda \to -\infty \).

\( r \to 0 \) means we do not penalize \( u \) much and as such, the optimal \( u \) can have large magnitude (i.e. control is cheap so we do not care about its power consumption, which is \( \approx \) magnitude\(^2\)). Since we do not have much restriction on \( u \), we can move the closed-loop pole as far to the left as we want while still minimize the cost function.

(b) \( r \to \infty \), then \( \lambda \to -|a| \).

\( r \to \infty \) means we do penalize \( u \) much and as such, we force the optimal \( u \) to have smaller and smaller magnitude (i.e. control is expensive so we want the control magnitude small). As control is more and more expensive, if the original system is unstable \( (a > 0) \), the best we can do is to stabilize the system with closed-loop pole at \(-|a|\) (in order to minimize the cost function; other stabilizing control will increase the cost). If the original system is already stable \( a < 0 \), the feedback gain is 0.
4. Another infinite horizon optimal problem

Here, \( Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), \( R = 1 \) and the ARE equation is

\[
A^T P + PA + Q - PBR^{-1}B^T P = \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} - P \begin{bmatrix} 0 \\ 1 \end{bmatrix} P = 0
\]

Let \( P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \), then

\[
\begin{bmatrix} -3p_2 & -3p_3 \\ p_1 + 2p_2 & p_2 + 2p_3 \end{bmatrix} + \begin{bmatrix} -3p_2 & p_1 + 2p_2 \\ -3p_3 & p_2 + 2p_3 \end{bmatrix} - \begin{bmatrix} p_2 & p_2p_3 \\ p_2p_3 & p_3 \end{bmatrix} = 0
\]

which reduces to

\[
-6p_2 - p_2^2 = 0 \\
-3p_3 + p_1 + 2p_2 - p_2p_3 = 0 \\
2(p_2 + 2p_3) - p_3 = 0
\]

The first equation gives \( p_2 = 0 \) or \( p_2 = -6 \).

If \( p_2 = 0 \), then \(-3p_3 + p_1 = 0\) and \(4p_3 - p_3^2 = 0\). That gives \( p_3 = 0, p_1 = 0 \) or \( p_3 = 4, p_1 = 12 \).

If \( p_2 = -6 \), then \(3p_3 + p_1 - 12 = 0\) and \(4p_3 - p_3^2 - 12 = 0\), which has no solution (since \(4p_3 - p_3^2 - 4 = -(p_3 - 2)^2 \leq 0 \forall p_3\) and so \(4p_3 - p_3^2 - 12 \leq -8 < 0\)).

Therefore, the ARE has solutions \( P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) and \( P = \begin{bmatrix} 12 & 0 \\ 0 & 4 \end{bmatrix} \), which are both in the class of positive semi-definite matrices. The corresponding feedback are \( u_1^* = 0 \) and \( u_2^* = -\begin{bmatrix} 0 & 4 \end{bmatrix} x = -Kx \).

The control \( u_1^* \) is not a stabilizing control since the original system is unstable (eigenvalues \( 1 \pm \sqrt{2i} \)).

The control \( u_2^* \) is a stabilizing control since the closed-loop eigenvalues are \( \chi(A - BK) = -1 \pm \sqrt{2i} \).

Therefore, the control \( u = -\begin{bmatrix} 0 & 4 \end{bmatrix} x \) is the optimal control that minimizes the given cost in the class of stabilizing control (i.e. \( x(t) \to 0 \) as \( t \to 0 \)). No other stabilizing control gives a smaller cost.

If we do not restrict to the class of stabilizing control, then we can have a smaller cost function (with \( u_1^* = 0 \)).

Comment: When the solution of the ARE in the class of positive semi-definite matrices is not unique, then we need to check to see which solution gives the actual optimal control because the ARE is a necessary condition only (see also the comment on Problem 2). When \((A, B)\) is stabilizable and \((A, C)\) is detectable (where \(Q = C^T C\)), we know that the solution of the ARE in the class of positive semi-definite matrices is unique and hence, it’s optimal. If \((A, C)\) is not detectable or \((A, B)\) is not stabilizable, then extra care is needed (see the note pages 186-189).

5. Finite Horizon Optimal Feedback

This is a finite horizon LQR problem so the solution is \( u^*(x, t) = -R^{-1}B^T P(t)x \) where \( P(t) \)
is a solution in the class of positive semi-definite matrices of the Differential Riccati Equation (DRE):

\[-\dot{P} = Q + PA + A^T P - PBR^{-1}B^T P\]

with the boundary condition \(P(t_f) = M\).

For the problem in the homework, the DRE is

\[-\dot{p} = 4 - p^2, \quad p(1) = 1.\]  \hspace{1cm} (2)

subject to \(p(t) \geq 0 \forall t \in [0, 1]\). Solving the DRE:

\[
\frac{dp}{4-p^2} = -dt \Leftrightarrow \frac{dp}{2-p} + \frac{dp}{2+p} = -4dt
\]

\[
\Rightarrow -\ln |2-p| + \ln |2+p| = -4t + C
\]

\[
\Leftrightarrow \ln \left| \frac{2+p}{2-p} \right| = -4t + C \Rightarrow \left| \frac{2+p}{2-p} \right| = e^{-4t+C}
\]

At \(t = 1\), \(e^{-4+C} = 3 \Rightarrow C = 4 + \ln 3\). Therefore,

\[
\frac{2+p}{2-p} = \pm 3e^4e^{-4t} \Rightarrow p(t) = \frac{6e^4e^{-4t} - 2}{1 + 3e^4e^{-4t}}
\]

(the other \(p(t)\) does not satisfy \(p(1) = 1\); in solving (2), we have one way implication so need to check back to get the correct solution with the given boundary condition).

The optimal control is

\[
u^*(x, t) = -\frac{6e^4e^{-4t} - 2}{1 + 3e^4e^{-4t}} x(t).
\]

Comment: This problem illustrates a finite horizon LQR problem, in which you need to solve a differential equation (DRE) and take care of the boundary condition. For general linear systems, the DRE can be solved using the Hamiltonian matrix (as in section 10.4 of the notes) but the method to solve the DRE is not the emphasis of this problem; for simple (e.g. scalar) DRE, we can solve it directly.

6. Generalized LQR

The HJB equation:

\[-\frac{\partial V^0(x, t)}{\partial t} = \min_u \left\{ x^T R_1(t)x + 2x^T R_12(t)u + u^T R_2(t)u + \frac{\partial V^0(x, t)}{\partial x}(A(t)x + B(t)u + c(t)) \right\}.\]

Now, guessing \(V^0(x, t) = x^T P(t)x + 2k^T(t)x + g(t)\). Boundary condition

\[V^0(x(t_1), t_1) = x(t_1)^T P(t_1)x(t_1) + 2k^T(t)x(t_1) + g(t_1) = x^T(t_1)Q_f x(t_1) \quad \forall x(t_1)\]

which implies

\[P(t_1) = Q_f, \quad k(t_1) = 0, \quad g(t_1) = 0.\]  \hspace{1cm} (3)

Now,

\[
\frac{\partial V^0(x, t)}{\partial t} = x^T \dot{P}(t)x + 2k^T(t)x + \dot{g}(t), \quad \frac{\partial V^0(x, t)}{\partial x} = 2x^T P(t) + 2k^T(t).
\]
The HJB becomes
\[-(x^T \dot{P}(t)x + 2k^T(t)x + \dot{g}(t)) = \min_u \{ x^T R_1(t)x + 2x^T R_{12}(t)u + u^T R_2(t)u + 2(x^T P(t) + k^T(t))(A(t)x + B(t)u + c(t)) \}. \tag{4}\]

The expression inside the bracket in the above HJB equation can be rewritten as
\[u^T R_2(t)u + 2v^T u + s\tag{5}\]
where
\[v^T = x^T (R_{12}(t) + P(t)B(t)) + k^T(t)B(t)\]
\[s = x^T (R_1(t) + 2P(t)A(t))x + 2x^T (P(t)c(t) + A^T(t)k(t)) + 2k^T(t)c(t).\]

The expression (5) is quadratic in \(u\) and \(R_2 > 0\), so it is minimized when \(u^* = -R_2^{-1}(t)v\) and the minimum value is \(-v^T R_2^{-1}(t)v + s\). (To see this clearly, write (5) as \((u - u^*)^T R_2(t)(u - u^*) - v^T R_2^{-1}(t)v + s \geq -v^T R_2^{-1}(t)v + s \forall u\). Thus, the optimum control is
\[u^*(x, t) = -R_2^{-1}(t)[(R_{12}^T + B^T(t)P(t))x(t) + B^T(t)k(t)].\]

Now, to find the equation for \(P(t)\) and \(k(t)\), substituting \(u^*\) into the HJB equation (4), we obtain
\[-(x^T \dot{P}(t)x + 2x^T k(t) + \dot{g}(t)) = -v^T R_2^{-1}(t)v + s\]
\[= x^T M(t)x + 2x^T P(t) + q(t)\tag{6}\]
where
\[M(t) = R_1(t) + A^T(t)P(t) + P(t)A(t) - (R_{12}(t) + P(t)B(t))R_2^{-1}(B^T(t)P(t) + R_{12}^T(t))\]
\[p(t) = -(R_{12}(t) + P(t)B(t))R_2^{-1}B^T(t)k(t) + (P(t)c(t) + A^T(t)k(t))\]
\[q(t) = k^T(t)B(t)R_2^{-1}B^T(t)k(t) + 2k^T(t)c(t)\]

(we used the fact that \(k^T(t)x = x^T k(t)\) and \(2x^T P(t)A(t)x = x^T (P(t)A(t) + A^T(t)P(t))x.\))

Since (6) is true for all \(x\), we then have the differential equations for \(P(t), k(t), g(t)\):
\[-\dot{P}(t) = R_1(t) + A^T(t)P(t) + P(t)A(t) - (R_{12}(t) + P(t)B(t)R_2^{-1}(B^T(t)P(t) + R_{12}^T(t))\]
\[-\dot{k}(t) = -(R_{12}(t) + P(t)B(t))R_2^{-1}B^T(t)k(t) + (P(t)c(t) + A^T(t)k(t))\]
\[-\dot{g}(t) = k^T(t)B(t)R_2^{-1}B^T(t)k(t) + 2k^T(t)c(t)\]

with the boundary condition (3).

The minimum value of \(V\) is
\[V^*(x_0, t_0) = x(t_0)^T P(t_0)x(t_0) + 2k^T(t_0)x(t_0) + g(t_0).\]

**Comment:** This problem helps understanding how to arrive at the DRE for linear systems from the HJB equation and the role of the boundary conditions in the DRE. The result in this problem reduces to the standard DRE if \(c = 0\) and \(R_{12} = 0\) (in which case \(k = 0\) and \(g = 0\)).

5
7. LQR Control with Stability Margin

(a) The cost function in terms of $z$ and $v$:

$$V(v) = \int_0^\infty z^T Q z + v^T R v dt$$

and the dynamic is

$$\dot{z} = \alpha e^{\alpha t} x + e^{\alpha t} \dot{x} = \alpha e^{\alpha t} x + e^{\alpha t} (Ax + Bu) = (\alpha I + A)e^{\alpha t} x + Be^{\alpha t} u = (\alpha I + A)z + Bv.$$ 

(b) The optimal solution is $z^* = -R^{-1} B^T P v$, which is equivalent to $u^* = -R^{-1} B^T P x$ where $P$ is a solution in the class of positive semi-definite matrices to the modified ARE equation

$$(A^T + \alpha I)P + P(A + \alpha I) + Q - PBR^{-1}B^T P = 0 \tag{7}$$

or

$$A^T P + PA + 2\alpha P + Q - PBR^{-1}B^T P = 0. \tag{8}$$

(c) If $(A, B)$ is controllable, then $(A + \alpha I, B)$ is also controllable. That is because $\text{rank}[A + \alpha I - sI|B] = \text{rank}[A - rI|B]$, $r = s - \alpha$, and since $(A, B)$ is controllable, $\text{rank}[A - rI|B] = n \forall r$ and hence, $\text{rank}[A + \alpha I - sI|B] = n \forall s$, which implies that $(A + \alpha I, B)$ is controllable. Similarly, If $(A, C)$ is observable, then $(A + \alpha I, C)$ is also observable. Since $(A + \alpha I, B)$ is controllable and $(A + \alpha I, C)$ is also observable (where $Q = C^T C$), from (7), by Theorem 10.5.2 in the notes, we know that $P$ is positive definite and is unique and further, $(A + \alpha I) - BR^{-1}B^T P$ is Hurwitz. That means the real parts of eigenvalues of $A + \alpha I - BR^{-1}B^T P$ are strictly less than 0, and hence real parts of eigenvalues of $A_{cl} = A - BR^{-1}B^T P$ are strictly less than $-\alpha$ (for details on this, see Homework 5 problem 1).