1. The Transition Matrix

(Hint: Recall that the solution to the linear equation $\dot{x} = Ax + Bu$ is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$  

This formula is extremely important for linear systems and should be memorized.)

First solve for $x_2$.

$$\dot{x}_2 = tx_2 \Rightarrow dx_2/x_2 = t dt \Rightarrow \ln x_2(t) - \ln x_2(t_0) = \frac{1}{2}(t^2 - t_0^2)$$

$$\Rightarrow x_2(t) = e^{\frac{1}{2}(t^2-t_0^2)}x_2(t_0).$$

Substituting this into equation for $x_1$:

$$\dot{x}_1 = -x_1 + e^{-\frac{1}{2}t^2}x_2 = -x_1 + e^{-\frac{1}{2}t^2_0}x_2(t_0)$$

Since $x_2$ is not a function of $x_1$, we can regard it as the input $u(t)$. So with $A = -1$, $B = e^{-\frac{1}{2}t^2}x_2(t_0), u(t) = e^{-t}$, we apply (1) to obtain:

$$\Rightarrow x_1(t) = e^{-(t-t_0)}x_1(t_0) + \int_{t_0}^{t} e^{-(t-\tau)}e^{-\frac{1}{2}\tau^2}x_2(t_0)d\tau$$

$$= e^{-(t-t_0)}x_1(t_0) + e^{-t-\frac{1}{2}t^2_0}x_2(t_0)(t-t_0)$$

We have

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-(t-t_0)} & (t-t_0)e^{-\frac{1}{2}t^2_0} \\ 0 & e^{\frac{1}{2}(t^2-t_0^2)} \end{pmatrix} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}$$

and so, by definition, the state transition matrix is

$$\Phi(t, t_0) = \begin{pmatrix} e^{-(t-t_0)} & (t-t_0)e^{-\frac{1}{2}t^2_0} \\ 0 & e^{\frac{1}{2}(t^2-t_0^2)} \end{pmatrix}.$$ 

You can check that this matrix indeed satisfies

$$\frac{\partial}{\partial t} \Phi(t, t_0) = A(t)\Phi(t, t_0); \Phi(t_0, t_0) = I,$$

and thus, the solution of the system is $x(t) = \Phi(t, t_0)x(t_0)$.

2. Matrix Linear Differential Equation (10 points)

(Hint: The Leibnitz Integral Rule:

$$\frac{d}{dt} \int_{a}^{t} f(t, \tau)d\tau = f(t, t) + \int_{a}^{t} \frac{\partial f(t, \tau)}{\partial t} d\tau$$

for a constant $a$. 

First we check the initial condition \( X(t_0) = I \cdot X_0 \cdot I + 0 = X_0 \). If \( \Phi_i \) is the transition matrix of \( \dot{x} = A_i x \), then
\[
\frac{\partial \Phi_i(t, t_0)}{\partial t} = A_i \Phi_i(t, t_0).
\]

Then using the Leibniz rule:
\[
\dot{X}(t) = \left[ \frac{\partial}{\partial t} \Phi_1(t, t_0) \right] X_0 \Phi_2^T(t, t_0) + \Phi_1(t, t_0) X_0 \left[ \frac{\partial}{\partial t} \Phi_2^T(t, t_0) \right] + \int_{t_0}^{t} \Phi_1(t, \tau) F(\tau) \left[ \frac{\partial}{\partial t} \Phi_2^T(t, \tau) \right] d\tau + F(t).
\]

Hence \( X(t) \) as given is a solution to the original system.

3. Convergence of LTI systems (15 points)

(a) We use the Jordan form (see Homework 3 problem 4; notice that here we have \( e^{At} \) instead of \( e^A \) so you need to include the variable \( t \) accordingly). You can also calculate by other methods (e.g. using the Cayley-Hamilton theorem or inverse Laplace transform).

\[
e^{A_1 t} = P e^{t} P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}.
\]

\[
e^{A_2 t} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3e^t - 2e^{-t} & -3e^t + 3e^{-t} \\ 2e^t - 2e^{-t} & -2e^t + 3e^{-t} \end{pmatrix}.
\]

\[
e^{A_3 t} = P e^{t} P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & 3e^{-2t} - 3e^{-t} \end{pmatrix}.
\]
(b) 

\[ A_1 : \quad x(t) = e^{A_1 t}x(0) = \begin{pmatrix} t(x_2(0) - x_1(0)) + x_1(0) \\ t(x_2(0) - x_1(0)) + x_2(0) \end{pmatrix} \] 

\[ A_2 : \quad x(t) = e^{A_2 t}x(0) = \begin{pmatrix} e^t[3x_1(0) - 3x_2(0)] - e^{-t}[2x_1(0) - 3x_2(0)] \\ e^t[2x_1(0) - 2x_2(0)] - e^{-t}[2x_1(0) - 3x_2(0)] \end{pmatrix} \] 

\[ A_3 : \quad x(t) = e^{A_3 t}x(0) = \begin{pmatrix} e^{-t}[x_1(0) + tx_3(0)] \\ e^{-t}[x_1(0) + tx_3(0)] - 3e^{-t}x_3(0) \\ e^{-t}[x_1(0) + tx_3(0)] - x^t x_3(0) \end{pmatrix} \] 

(c) A_1 system: the solution stays bounded if \( x_1(0) = x_2(0) \), or goes to infinity otherwise. The subspace \( \text{span}\{(1, 1)^T\} \) is stable in the sense of Lyapunov, but it is not a stable subspace (just check the definition of stability subspaces).

A_2 system: if \( x_1(0) = x_2(0) \), the solution decays to 0, and goes to infinity otherwise. The \( \text{span}\{(1, 1)^T\} \) is its stable subspace corresponding to eigenvalue \(-1\).

A_3 system: asymptotically decays to 0 for all initial conditions.

(d) Let \( \lambda_i \) be the eigenvalues of the matrix \( A \).

i. The solution will decay to zero for arbitrary initial condition (i.e. the system is asymptotically stable) if and only if \( \text{Re}(\lambda_i) < 0 \forall i \).

ii. If \( \text{Re}(\lambda_i) \leq 0 \), and the eigenvalues with zero real parts have **linearly independent eigenvectors** (note: not generalized eigenvectors. Generalized eigenvectors are not eigenvectors), the system will be bounded in the sense of Lyapunov.

iii. Otherwise, the system is unstable (it can go to infinity for some initial states). If there exists some \( \text{Re}(\lambda_i) < 0 \), their corresponding eigenvectors will span an asymptotically stable subspace. If \( \text{Re}(\lambda_i) > 0 \) for all \( i \), the system is unbounded for all nonzero initial states.

To sum up, these are things one need to remember:

i. \( \text{Re}(\lambda) < 0 \iff x_0 \in \text{span}(\text{eigenvectors, gen. eigenvectors}) \rightarrow \text{asymptotic stability} \)

ii. \( \text{Re}(\lambda) = 0 + \) have only eigenvectors, not gen. eigenvectors
   \( \iff x_0 \in \text{span}(\text{eigenvectors}) \rightarrow \text{stability (but not asymptotic)} \)

iii. \( \text{Re}(\lambda) = 0 + \) have gen. eigenvectors
   \( \iff x_0 \in \text{span}(\text{eigenvectors, gen. eigenvectors}) \rightarrow \text{instability} \)

iv. \( \text{Re}(\lambda) > 0 \iff x_0 \in \text{span}(\text{eigenvectors, gen. eigenvectors}) \rightarrow \text{instability} \)

4. Linear Time-Varying System with Negative Eigenvalues (10 points) 
   For the given LTV system:

   (a) Similar to problem 1, we solve for \( x(t) \).

   \[ \dot{x}_2 = -x_2 \Rightarrow x_2(t) = e^{-(t-t_0)}x_2(t_0). \]
Then solve for $x_1(t)$:

$$
\dot{x}_1 = -x_1 + e^{2t}x_2
$$

$$
\Rightarrow x_1(t) = e^{-(t-t_0)}x_1(t_0) + \int_{t_0}^{t} e^{-(t-\tau)}e^{2\tau}e^{-(\tau-t_0)}x_2(t_0)d\tau
\]

$$
= e^{-(t-t_0)}x_1(t_0) + \frac{1}{2}(e^{t+t_0} - e^{-t+3t_0})x_2(t_0).
$$

And thus,

$$
\begin{pmatrix}
 x_1(t) \\
 x_2(t)
\end{pmatrix}
= \begin{bmatrix}
 e^{-(t-t_0)} & \frac{1}{2}(e^{t+t_0} - e^{-t+3t_0}) \\
 0 & e^{-(t-t_0)}
\end{bmatrix}
\begin{pmatrix}
 x_1(t_0) \\
 x_2(t_0)
\end{pmatrix}
$$

and hence, the state transition matrix is

$$
\Phi(t, t_0) = \begin{bmatrix}
 e^{-(t-t_0)} & \frac{1}{2}(e^{t+t_0} - e^{-t+3t_0}) \\
 0 & e^{-(t-t_0)}
\end{bmatrix}.
$$

(b) The solution to the differential equation with $x(0) = [0, 1]^T$ is:

$$
x(t) = \Phi(t, 0)x(0) = \begin{bmatrix}
 \frac{1}{2}(e^t - e^{-t}) \\
 e^{-t}
\end{bmatrix}.
$$

(c) The eigenvalues of $A(t)$ are $-1, -1$ for all $t \geq 0$. However, since $A$ is *time varying*, this does not necessarily imply that the system is stable.

(d) As $t \to \infty$, the first component of $x$ above becomes unbounded, whereas the second component goes to zero. This may at first seem surprising since both eigenvalues of $A$ are negative, but not on a second thought since $A$ is time-varying. Also, this is inline with part (c).

5. **Solution of Linear Systems** (10 points)

We first find the solution of the system. Eigenvalues of $\begin{bmatrix}
 0 & 1 \\
 -4 & -5
\end{bmatrix}$ are $-1, -4$ and the corresponding eigenvectors are $(1, -1)^T, (1, -4)^T$. Then

$$
x(t) = e^{\begin{bmatrix}
 0 & 1 \\
 -4 & -5
\end{bmatrix}t}x(0) = \begin{bmatrix}
 1 & 1 \\
 -1 & -4
\end{bmatrix} e^{\begin{bmatrix}
 -1 & 0 \\
 0 & -4
\end{bmatrix}t} \begin{bmatrix}
 1 & 1 \\
 -1 & -4
\end{bmatrix}^{-1} x(0)
$$

$$
= -\frac{1}{3} \begin{bmatrix}
 -4e^{-t} + e^{-4t} & -e^{-t} + e^{-4t} \\
 4e^{-t} - 4e^{-4t} & e^{-t} - 4e^{-4t}
\end{bmatrix} x(0) = -\frac{1}{3} \left( (4x_{0_1} + x_{0_2})e^{-t} - 4(x_{0_1} + x_{0_2})e^{-4t} \right)
$$

and hence,

$$
y(t) = \begin{bmatrix}
 1 & 1
\end{bmatrix} x(t) = (x_{0_1} + x_{0_2})e^{-4t}.
$$

Therefore, if $x(0) \in \text{span}\{\begin{bmatrix}
 1 \\
 -1
\end{bmatrix}\}$, the mode corresponding to $\lambda = -4$ does not appear at the output i.e. it’s guaranteed that only the mode corresponding to the eigenvalue $-1$ is excited at the output (which is zero in this case).
**Another solution:** Eigenvalues $\lambda_1 = -1, \lambda_2 = -4$ and eigenvectors $v_1 = (1, -1)^T, v_2 = (1 - 4)^T$, correspondingly. Since the eigenvectors are linearly independent, for any $x_0$, we have $x_0 = \alpha v_1 + \beta v_2$ for some $\alpha, \beta$. Then

$$x(t) = e^{At}x_0 = e^{At}(\alpha v_1 + \beta v_2) = \alpha e^{At}v_1 + \beta e^{At}v_2.$$  

Using the fact that $Av_1 = \lambda_1 v_1$ and using Taylor expansion of $e^{At}$, we can show that $e^{At}v_1 = e^{\lambda_1 t}v_1$. Similarly, $e^{At}v_2 = e^{\lambda_2 t}v_2$. Hence

$$x(t) = \alpha e^{\lambda_1 t}v_1 + \beta e^{\lambda_2 t}v_2,$$

and so,

$$y(t) = Cx(t) = \alpha e^{\lambda_1 t}Cv_1 + \beta e^{\lambda_2 t}Cv_2,$$

We have $Cv_2 \neq 0$ so in order for $e^{\lambda_2 t}$ not appearing at the output, we must have $\beta = 0$. Thus, $x_0$ is of the form $\alpha v_1$, which means $x_0 \in \text{span}\{v_1\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

6. **Linear Time-Varying Systems with (Skew) Symmetric Matrix $A(t)$** (10 points)

(a) Choose a Lyapunov function candidate $V(x, t) = x^T x$. It’s clear that $V$ is positive definite i.e $V(x) \geq 0$ for all $x$ and $V(x) = 0 \Rightarrow x = 0$. Then:

$$\dot{V}(x, t) = x^T A^T x + x^T Ax = x^T (A^T + A)x = 0.$$

The last equation uses the fact that $A$ is skew symmetric. Since $\dot{V}(x, t) \leq 0$, the system is stable. It is uniformly stable because $V$ does not depend explicitly on $t$.

(b) Again choose a Lyapunov function candidate $V(x, t) = x^T x$. It’s clear that $V$ is positive definite i.e $V(x) \geq 0$ for all $x$ and $V(x) = 0 \Rightarrow x = 0$. Then:

$$\dot{V}(x, t) = x^T A^T x + x^T Ax = 2x^T Ax \leq -2x^T (\epsilon I)x = -2\epsilon x^T x < 0 \forall x \neq 0.$$

Hence the system is uniformly asymptotically stable (it is exponentially stable in this case). Uniformity follows from the fact that $V$ does not depend explicitly on $t$ (it’s a function of $x$ only).

7. **Periodic linear systems** (10 points)

(a) We have

$$P^{-1}(t)e^{R(t-t_0)}P(t_0) = \Phi(t, 0)e^{-Rt} \cdot e^{R(t-t_0)} \cdot e^{Rt_0} = \Phi(t, 0)\Phi^{-1}(t_0, 0) = \Phi(t, t_0).$$

Notice that for matrices, $(AB)^{-1} = B^{-1}A^{-1}$.

(b) Since $x(t) = \Phi(t, 0)x(0)$,

$$\ddot{x}(t) = P(t)x(t) = e^{Rt}\Phi^{-1}(t, 0)\Phi(t, 0)x(0) = e^{Rt}x(0).$$

Therefore, $\ddot{x} = Re^{Rt}x(0) = R\ddot{x}$. 

5
(c) By the definition of the transition matrix,
\[ \dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0). \]

Letting \( t_0 = 0 \) and \( t_0 = T \), we have the following two differential equations:
\[
\begin{align*}
\dot{\Phi}(t, 0) &= A(t)\Phi(t, 0) \\
\dot{\Phi}(T + t, T) &= A(t + T)\Phi(T + t, T)
\end{align*}
\]
Since \( A(t) \) to be periodic, \( A(t) = A(t + T) \forall t \), and we then have
\[
\begin{align*}
\dot{\Phi}(t, 0) &= A(t)\Phi(t, 0) \\
\dot{\Phi}(T + t, T) &= A(t)\Phi(T + t, T)
\end{align*}
\]
Since \( \Phi(T, T) = \Phi(0, 0) = I \), which means the boundary conditions for \( t = 0 \) are also the same, we have that \( \Phi(T + t, T) \) and \( \Phi(t, 0) \) are the state transition matrices of the same equation \( \dot{x} = A(t)x \). By the uniqueness of solutions of differential equations, we must have
\[ \Phi(T + t, T) = \Phi(t, 0). \]

Note: We do not have a close form of \( \Phi(t, t_0) \) for a time-varying \( A(t) \) in general, but only the Peano-Baker series expansion of it (page 56, ECE515 course note).

(d) Property of the transition matrix: \( \Phi(t + T, 0) = \Phi(T + t, T)\Phi(T, 0) \) and so, \( \Phi^{-1}(t + T, 0) = \Phi^{-1}(T, 0)\Phi^{-1}(T + t, T) \). We then have
\[
\begin{align*}
P(t + T) &= e^{R(t+T)}\Phi^{-1}(t + T, 0) = e^{Rt}e^{RT} \cdot \Phi^{-1}(T, 0)\Phi^{-1}(t + T, T) \\
&= e^{Rt}\Phi^{-1}(t + T, T),
\end{align*}
\]
since \( \Phi(T, 0) = e^{RT} \) is given by the problem. Since \( \Phi(t, t_0) \) is periodic with period \( T \), \( \Phi^{-1}(t, t_0) \) is also periodic with period \( T \). Thus,
\[ P(t + T) = e^{RT}\Phi^{-1}(t, 0) = P(t). \]