1. **Stability Margin of Linear Systems** (10 points)

   Show that all eigenvalues of $A$ have real parts less than $-\alpha < 0$ if and only if, for any given positive definite symmetric matrix $Q$, the equation:

   $$A^TP + PA + 2\alpha P = -Q$$

   has a unique (symmetric) positive definite solution $P$. (Problem 5.18 from Chen). The scalar $\alpha$ sometimes is called (an estimate of) the stability margin of the LTI system $\dot{x} = Ax$. Show that for such a system, $\|e^{\alpha t}x(t)\| \to 0$ as time $t \to \infty$.

   **Hint:** First explain why $A + \alpha I$ has all eigenvalues with real part $< 0$, then use the Lyapunov theorem. Can you write $e^{\alpha t}x(t)$ as a solution of some asymptotically stable system?

2. **Stability of a Nonlinear System** (10 points)

   Consider the nonlinear system:

   $$\begin{align*}
   \dot{x}_1 &= -x_1 + x_2 \\
   \dot{x}_2 &= x_1 - x_2 - x_3^2
   \end{align*}$$

   (a) Find all the equilibrium states.

   (b) Linearize the system around each equilibrium and determine whether the linearized system is stable or not using the eigenvalue test. From the stability of the linearized system, what can you say about the local stability of the original nonlinear system?

   (c) Study stabilities of each of the equilibrium of the original nonlinear system using Lyapunov method.

3. **Controllability of LTI Systems** (10 points)

   Given the linear time-invariant system:

   $$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u := Ax + Bu.$$

   (a) Check the controllability using

   i. the controllability matrix

   ii. rows of $\bar{B} = M^{-1}B$, where $M$ is chosen such that $\bar{A} = M^{-1}AM$ is diagonal.

   iii. the Hautus-Rosenbrock test

   (b) Identify the controllable and uncontrollable modes of the system, its controllable and uncontrollable subspaces, and convert the system to a Kalman canonical form.

   **Hint:** Read the textbook page 158-164.

   (c) Suppose that we start from the initial state $x(0) = (1, 1, -1)^T$. Is there a control $u(t)$ that drives the state to $x(1) = (3, -1, 1)^T$ at time $t = 1$? Is there a control $u(t)$ that drives the state to $x(1) = (-1, 1, 1)^T$ at time $t = 1$?
4. Controllability and observability of LTV Systems (10 points)

Check controllability and observability of
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]
\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} x
\]

Hint: Read the textbook Section 6.8. Note that here we work with a LTV system, not LTI systems.

5. Duality of Controllability and Observability of Linear Systems (10 points)

(a) For LTI systems, show that \((A, B)\) is controllable if and only if \((-A, B)\) is controllable. Note that this is not true for time-varying systems.

(b) For LTI systems, show that \((A, B)\) is controllable if and only if \((-A^T, B^T)\) is observable.

Comment: This property is called controllability/observability duality. It is also true for time-varying systems i.e. \(A(t), B(t)\) is controllable if and only if \(-A(t)^T, B(t)^T\) is observable. For time varying systems, the proof is more involved. In the light of part a), is this true that \(A(t), B(t)\) is controllable if and only if \(A(t)^T, B(t)^T\) is observable?

6. Output Controllability (10 points)

Given a linear time invariant system:
\[
\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, u \in \mathbb{R}^n;
\]
\[
y = Cx \quad y \in \mathbb{R}^n
\]

it is said to be output controllable if, for some \(t_1 > t_0\), for all \(y_1 \in \mathbb{R}^n\), there exists a control \(u(\cdot)\) which drives the output from \(y_0 = Cx_0\) to \(y_1\). Similar to the proof for state controllability, prove that:

(a) The system is output controllable iff for some \(t_1 > t_0\),
\[
CW_c(t_1 - t_0)C^T > 0
\]

where \(W_c(t_1 - t_0)\) is the controllability grammian.

(b) The system is output controllable iff
\[
\text{Rank}(CC) = n_o = \dim(y)
\]

where \(C\) is the controllability matrix.

Hint: The problem is not as difficult as it seems. Consult the proof of Theorem 5.2.2 in the course note. Examine the relationship among positive definiteness and nonsingularity of a symmetric matrix.