Problems:

1. **Linear vs. Nonlinear Maps** (10 points)
   In this problem, we will revisit the concept of linear/nonlinear map.

   Suppose $A, B, C, X \in \mathbb{C}^{n \times n}$. Consider the following maps from $\mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ and determine if they are linear or not. If the map is linear, show why it is true and provide a counterexample if it is false:

   (a) $X \mapsto AX + BXC$
   (b) $X \mapsto AXA - B$
   (c) $X \mapsto AX + XBX$

   Hint: Using the definition of what a linear map is.

2. **Inner Product** (10 points)
   The concept of inner product leads to the concept of norm, which is a generalization of a measure of a length.

   Let $(V, \mathbb{C})$ be an inner product space.

   (a) Let $x, y \in V$ with $x$ orthogonal to $y$. Prove the Pythagorean theorem:

   $$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

   Hint: Recalled that by definition, $x, y$ are orthogonal if $\langle x, y \rangle = 0$ where $\langle \cdot, \cdot \rangle$ is the inner product. Then proceed using the properties of an inner product.

   (Comment: This is exactly the same as the Pythagorean theorem you learn in high school except that it is applicable to general inner product spaces, not just $\mathbb{R}^2$ but any inner product spaces (such as spaces of functions). You may want to try to write down the Pythagorean theorem for functions to see what it looks like.)

   (b) Prove the Parallelogram law, that is in an inner product space,

   $$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

   Can you draw a geometric explanation of this in $\mathbb{R}^2$?

3. **Jordan Form** (10 points)
   For a given matrix, the Jordan form is important because its power can be easily computed (which in turns is useful in calculation of solutions of linear systems).
(a) Show how to use similarity transformation to transform the following matrix into a Jordan form

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
-1 & 3 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

Specifically, construct the matrix \( P \) from the eigenvectors/generalized eigenvectors such that \( P^{-1}AP = D \) where \( D \) is the Jordan form of \( A \).

(b) We now calculate some matrix power and matrix exponential using Jordan form. For an integer \( m \), find the \( A^m \) for each of the following matrix in Jordan form

\[
A = \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}, \quad A = \begin{bmatrix}
a & 1 \\
0 & a
\end{bmatrix}
\]

Hint: Just do a few multiplication and you can see the pattern. Then prove the formula for \( A^m \) by induction.

(c) Now, for a given matrix \( A \), we have

\[
e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots.
\]

Use the result in part (b) to calculate \( e^A \) for each of the matrix in part (b).

(d) Now, if \( A \) is not in Jordan form, we can use similarity transformation to get its Jordan form such that \( P^{-1}AP = D \), and so \( A = PDP^{-1} \) where \( D \) is a Jordan form of \( A \). Using this, first show that \( e^A = Pe^DP^{-1} \), then use part (b) and (c) to calculate \( e^A \) for the matrix \( A \) in part (a).

4. Invariant Subspaces (10 points)

In this problem, we will learn a little bit deeper into the reasoning behind Jordan block.

Let \( L : V \to V \) be a linear map of a \( n \)-dimensional vector space \( V \) over the field \( F \). A subspace \( M \subset V \) is called \( L \)-invariant if \( L(x) \in M \) for all \( x \in M \). Suppose \( V \) is a direct sum of two subspaces \( M_1 \) and \( M_2 \), i.e., \( M_1 \cap M_2 = \{0\} \) and \( M_1 + M_2 = V \). If both \( M_1 \) and \( M_2 \) are \( L \)-invariant, show that there exists a matrix representation \( A \in F^{n \times n} \) of \( L \) of the form:

\[
A = \begin{bmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{bmatrix},
\]

with \( \text{dim}(A_{11}) = \text{dim}(M_1) \) and \( \text{dim}(A_{22}) = \text{dim}(M_2) \).

(Comment: The whole idea of Jordan form for a matrix \( A \) is basically decomposing the space \( \mathbb{C}^n \) into a direct sum of the \( A \)-invariant generalized eigenspaces of \( A \).)

5. Application of Cayley-Hamilton Theorem (10 points)

Given the matrix:

\[
A = \begin{bmatrix}
-1 & 2 & 0 \\
1 & 1 & 0 \\
2 & -1 & 2
\end{bmatrix},
\]

use the Cayley-Hamilton Theorem to compute:
(a) Its inverse $A^{-1}$.
(b) $A^6$.

6. **Calculation of $e^{At}$** (10 points)
Calculation of $e^{At}$ is important because this is what we use in the solution of a linear system $\dot{x} = Ax + Bu$. There are many ways to calculate $e^{At}$. In fact, there is a paper entitled “Nineteen Dubious Ways to Compute the Exponential of a Matrix” by Cleve Moler, Charles Van Loan in Siam Review, Volume 20, Issue 4, 1978 (a pdf file is on the class web site). In problem 4, we have used the Jordan form to calculate $e^{At}$. We will now demonstrate two other popular methods. Consider a matrix

\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
\]

(a) Using the Cayley-Hamilton theorem: Using the spectrum method described in page 64 of the text book, calculate $e^{At}$.
(b) Using Laplace transform method: Note that $\mathcal{L}(e^{At}) = (sI - A)^{-1}$ (see textbook page 70). First calculate $(sI - A)^{-1}$, then use inverse Laplace transform to calculate $e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$.

You need to show all calculations in reaching the results.

7. **Correlation between transfer functions and state-space equations** (10 points)
In this problem, we reexamine state space representations, and see how they appear in the frequency domain. A general state-space description of a LTI system is

\[
\dot{x} = Ax + Bu,
\]
\[
y = Cx + Du.
\]

(a) Assume zero-initial condition, first transform the above equation into frequency domain using Laplace transformation (e.g. $\mathcal{L}(\dot{x}) = sX(s)$), and then obtain a transfer-function expression of the system, i.e., derive $G(s)$ where $Y(s) = G(s)U(s)$. Note that in general, for MIMO system, $G(s)$ is a matrix whose elements are transfer functions.

(b) A flying aircraft is a dynamical system. Generally, they can be written as a nonlinear state-space equation of the form $\dot{x} = f(x, u)$ (these equations are derived from physical laws). For example, under certain operational conditions, a simplified equation of longitudinal axis dynamic is

\[
\dot{V} = \frac{D}{M} - g \sin \gamma + \frac{T}{M},
\]
\[
V \dot{\gamma} = \frac{L}{M} - g \cos \gamma + \alpha \frac{T}{M}
\]

where the state variables are the speed $V$ and the flight path angle $\gamma$, and the input is the thrust $T$; all others symbols are constant ($D$ and $L$ are the drag and lift, $M$ is the mass, $\alpha$ is the angle of attack). Let the output be $y = \begin{pmatrix} V \\ \gamma \end{pmatrix}$. Using the result in part (a), obtain a transfer function of the linearized aircraft around the operating point $V = \bar{V}$, $\gamma = 0$, $T = \bar{T}$.

Hint: first linearize the system around the given operating point (note to divide $V$ in the second equation first), then write down the transfer function.