Important Facts in Linear Algebra

Let $A \in \mathbb{C}^{n \times n}$. Denote its conjugate transpose $A^T = A^*$. Suppose $A$ has $p$ distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, with the $i$th one being of multiplicity $m_i$. The set of all eigenvalues of $A$ is called its spectrum, denoted as $\sigma(A)$. Let the characteristic polynomial of $A$ be

$$
\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0.
$$

Then we have the following:

1. $\det(A) = (-1)^n c_0 = \prod_{i=1}^{p} \lambda_i^{m_i}$. Furthermore, $\det(A) = \det(A^T)$. $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{C}^{n \times n}$.

2. $\text{Trace}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{p} m_i \lambda_i = (-1)^{n-1} c_{n-1}$. $\text{Trace}(A + B) = \text{Trace}(A) + \text{Trace}(B)$.

3. If $\lambda$ is an eigenvalue of $A$, its complex conjugate $\bar{\lambda}$ is an eigenvalue of $A^*$.

4. If $A$ is real, so are $c_i$’s. Therefore, if $\lambda$ is an eigenvalue, so is $\bar{\lambda}$.

5. If $A$ is both real and symmetric, then $\sigma(A)$ is real.

6. In general, $A$ is called Hermitian if $A = A^*$. For real matrix, this is synonymous to being symmetric. For a Hermitian matrix $A$, $\sigma(A)$ is real, and it admits a complete set of $n$ orthogonal eigenvectors (even if the eigenvalues are not distinct). Denote the normalized versions of these eigenvectors by $x_1, \ldots, x_n$ where $x_i$ corresponds to the eigenvalue $\lambda_i$ where we allow the possibility for $\lambda_i = \lambda_j$ for $i \neq j$. Then we can rewrite $A$ as:

$$
A = \sum_{i=1}^{n} \lambda_i x_i x_i^*
$$

which is known as the spectral representation of $A$.

7. If $A$ is not Hermitian, but semisimple (i.e., has a set of $n$ linearly independent eigenvectors $x_i$’s), it still admits a spectral representation, this time of the form

$$
A = \sum_{i=1}^{n} \lambda_i x_i y_i^*
$$

where $y_i^*$ is the $i$th row of $M^{-1}$, with $M = [x_1, \ldots, x_n]$. This is known as the eigenvector dyadic expansion of $A$.

8. If a Hermitian matrix $A$ has only positive (respectively, nonnegative) eigenvalues, it is called a positive definite (respectively, nonnegative definite) matrix, and this property is symbolically displayed as $A > 0$ (respectively, $A \geq 0$). $A$ is said to be negative definite (respectively, nonpositive definite) if $-A > 0$ (respectively, $-A \geq 0$). A positive definite matrix $A$ has the property that $x^* A x > 0$ for all $x \in \mathbb{C}^n$ which is not zero.