For the system \( \dot{x} = Ax + Bu \), assume \( C \in \mathbb{R}^{m \times n} \) and \( \text{rank}(C) = m < n \). (otherwise \( x = (C^T C)^{-1} y \).)

Define new state
\[
\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} y \\ \bar{x}_1 \end{bmatrix} = \begin{bmatrix} C \\ \bar{C} \end{bmatrix} x
\]
\[
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u
\]
\[
y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \bar{u}
\]
\[
\dot{\bar{x}}_2 = A_{22} \bar{x}_2 + (A_{21} y + \bar{B}_2 u)
\]
\[
\Rightarrow \dot{y} - A_{11} y = (A_{12} \bar{x}_2 - \bar{B}_1 u) = \text{known}
\]

So about the unknown state \( \bar{x}_2 \)
\[
\dot{\bar{x}}_2 = A_{22} \bar{x}_2 + \bar{u}
\]
\[
\dot{y} = A_{12} \bar{x}_2
\]

The pair \((A_{22}, A_{12})\) must be observable since \((A, [I, 0])\) is observable one of the midterm problem 1 (d).

So we can build an observer for \( \bar{x}_2 \) just like above, it is of \( n - m \) dimension.
What we have known about (static) feedback so far:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= cx
\end{align*}
\]
For state-feedback, if \((A, B)\) is controllable, we may place the poles of the transfer function
\[
H_c(s) = C (sI - (A + BK))^{-1} B
\]
at wherever we want. Nonetheless, the zeros are in general not affected at all.

As we know already, the output feedback is usually more restricted than the state feedback. Even if the system \(\dot{x} = Ax + Bu, \ y = cx\) is completely controllable and observable, we in general could not arbitrarily assign the system poles. Except in the trivial case that
\[
\text{rank}(C) = n \implies \text{reduce to state feedback}
\]

For example, as in handout #6, for a single input multiple \((p)\) output \((q)\) controllable/observable system.
\[
\begin{align*}
\dot{x} &= Ax + Bu, \ b \in \mathbb{R}^n \\
y &= cx, \ C \in \mathbb{R}^{pxn}, \ \text{rank}(C) = p < n
\end{align*}
\]
After output feedback
\[
\dot{x} = (A + bk^T C) x + bv \\
y = cx
\]
we may only assign \(\phi\) at most \(p\) poles. Why?
\[ X_A + bk^Tc(s) = s^n + \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_1s + \beta_0. \]

\[ \beta_i, i = 0, \ldots, n-1 \text{ are functions of } k = (k_1, \ldots, k_p)^T. \]

We only have \( p \) free parameters to tune \( n \) coefficients which is not enough.

However, in practice, the output \( y = cx \) is what we actually know about the system states. Direct state feedback is usually not realistic; on the other hand, direct output feedback is too limited, it does not allow us to assign the number of poles more than the number of outputs.

What is the way out? Instead of using static feedback, we use dynamic feedback. In other words, we try to design a pseudo-state feedback \( u = K\hat{x} \),

where \( \hat{x} \) is the estimated state (from the output and other things we know about the system) \((A, B, C \text{ known})\). Otherwise - adaptive control (417).
Is it possible to design a gain matrix $L$, using the information about the difference in the outputs $y - \hat{y} = y - c\hat{x}$, s.t.
\[
|\hat{x}(t) - x(t)|^2 \to 0 \quad \text{at } t \to \infty \quad \forall u?
\]
If so, we get ourselves an observer!
\[
\begin{aligned}
\dot{x} &= Ax + Bu, \quad y =Cx \\
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = c\hat{x}
\end{aligned}
\]
Define the "error state"
\[
\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}
\]
\[
\begin{aligned}
\dot{e} &= \dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + L(y - \hat{y}) \\
&= (A - LC)e
\end{aligned}
\]
The system is converted to
\[
\begin{aligned}
\begin{bmatrix} x \\ \hat{x} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\
y &= Cx.
\end{aligned}
\]
Under what conditions that

i) we can place the poles of $A-LC$ anywhere we want?

\[\text{\circ} \quad (A, C) \text{ observable } \iff (A^T, C^T) \text{ controllable} \]

ii) we can find $L$ s.t. $A-LC$ is stable, i.e. $e \to 0$?

\[\text{\circ} \quad (A, C) \text{ detectable } \iff (A^T, C^T) \text{ stabilizable} \]

If $\dot{e} = (A-LC)e$ is stable, we call the dynamical system
\[
\begin{aligned}
\dot{x} &= A\hat{x} + Bu + L(y - c\hat{x})
\end{aligned}
\]
an observer for the original system. It asymptotically recovers the state of the original system, although it does not have to start at the same initial state $x(0)$.
Consider a static linear feedback law of the estimated state:

\[ u = k \hat{x} + v = k(x - e) + v \]

\[ \dot{x} = Ax + Bu = A(x + B(k(x - e) + v)) \]

\[ = (A + BK)x - BKe + Bu. \]

\[ \Rightarrow \begin{cases} 
\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\
\end{cases} \]

\[ y = \begin{bmatrix} C \\ 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \]

\[ \det(sI - Ac) = \det(sI - (A+Bk)) \det(sI - (A-LC)) \Rightarrow 6(Ac) = 6(A+Bk) U 6(A-LC) \]

\((A, B)\) controllable \(\Rightarrow 6(A+Bk)\) arbitrary.

\((A, C)\) observable \(\Rightarrow 6(A-LC)\) arbitrary.

So \((A, B, C)\) is both controllable and observable implies that the poles of the system can be arbitrarily placed by dynamic feedback.

Stability of the observer system under the dynamical feedback:

\[ \hat{x} = \begin{bmatrix} 0 & A \end{bmatrix} \hat{x} + Bu + L(y - \hat{y}) \]

\[ \hat{y} = C\hat{x} \]

\( u = k\hat{x} \)

Stable?
A+Bk stable and A-LC stable does not imply that A+Bk-LC has to be stable!

To see this:

\[ A = -2 \quad B = 1 \quad K = 1 \quad C = 1 \quad \delta = -3/2 \]
\[ A+Bk = -2 + 1 = -1 \quad \text{stable} \]
\[ A-LC = -2 + 3/2 = -1/2 \quad \text{stable} \]

However,
\[ A+Bk-LC = -2 + 1 + 3/2 = 1/2 \quad \text{unstable} \]

So in general, looking at the observer separately, it may not be a stable (input-output) system.

Reduced order observer.

The order of the observer system is always n, regardless of the number of the outputs! unreasonable

See.

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \]
\[ C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

in this case \[ x_3 = y_1 - y_2 \] no need to estimate

We only need to estimate (x_1, x_2) at most!

In fact, we only need an observer of order n-rank(C).