1. **Characteristic functions** (10 points)
   Find the means and variances of random variables with the following characteristic functions:
   \[ \phi(\omega) = \left[ \lambda/(i\omega + \lambda) \right] \exp(-\tau i \omega), \]
   \[ \phi(\omega) = \exp(-i \omega / \lambda), \]
   \[ \phi(\omega) = \left[ \lambda/(i\omega + \lambda) \right]^2, \]
   (1)
   where \( \tau, \lambda > 0 \) and \( i \equiv \sqrt{-1}. \)

2. **Maximum and minimum statistics** (10 points)
   If \( X \) and \( Y \) are independent random variables, show that \( U = \min\{X, Y\} \) and \( V = \max\{X, Y\} \) have distribution functions
   \[ F_U(u) = 1 - [1 - F_X(u)][1 - F_Y(u)], \quad F_V(v) = F_X(v)F_Y(v). \]  
   (2)
   Now let \( X \) and \( Y \) be independent exponential random variables, with parameter \( \lambda = 1 \), show that
   (a) \( U \) is also exponential, with parameter \( \lambda = 2. \)
   (b) \( V \) has the same distribution as \( X + \frac{1}{2} Y. \) Hence find the mean and variance of \( V. \)

3. **Order statistics** (20 points)
   Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed variables with a common density function \( f. \) Such a collection is called a random sample. For each \( \omega \in \Omega \), arrange the sample values \( X_1(\omega), \ldots, X_n(\omega) \) in non-decreasing order \( X_{(1)}(\omega) \leq X_{(2)}(\omega) \leq \cdots \leq X_{(n)}(\omega) \), where \( (1), (2), \ldots, (n) \) is a permutation of \( 1, 2, \ldots, n \) (according to the order of values of \( X_i \)'s). The new variables \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) are called the order statistics.
   (a) Show, by a symmetry argument, that the joint distribution function of the order statistics:
   \[ P(X_{(1)} \leq y_1, \ldots, X_{(n)} \leq y_n) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} L(x_1, \ldots, x_n)n!f(x_1)\cdots f(x_n)dx_1\cdots dx_n \]
   where \( L = 1 \) if \( x_1 < x_2 < \cdots < x_n \) and \( L = 0 \) otherwise. Deduce that the joint density function of \( X_{(1)}, \ldots, X_{(n)} \) is \( n!L(y_1, \ldots, y_n)f(y_1)\cdots f(y_n). \)
   (b) Find the marginal density function of the \( k^{th} \) order statistic \( X_{(k)} \) by integrating the result of (a)
   (c) Find the marginal density function of \( X_{(k)} \) directly.
   (d) Now further assume that \( X_1, \ldots, X_n \) are uniform distributions on \([0, 1]\). Show that for fixed \( k \), the density function of \( Y = nX_{(k)} \) converges as \( n \to \infty \), and find this limit function.
4. **Construct examples** (10 points)

(a) Give an example of two random variables $X, Y$ such that $E[Y] = \infty$ but $E[Y|X]$ is finite.

(b) Give an example of two random variables $X, Y$ which are un-correlated, i.e. $Cov(X, Y) = 0$, but they are *not* independent.

5. **Approximation of a r.v. by a constant number** (10 points)

Given a random variable $X$, prove the following identity:

$$E[(X - c)^2] = Var(X) + (c - E[X])^2, \quad \forall c \in \mathbb{R}. \quad (3)$$

Hence if we like to “approximate” $X$ by a constant $c \in \mathbb{R}$ such that the following mean squared error $E[(X - c)^2]$ in minimized, the optimal $c$ is $E[X]$. [Notes: In this question and what below, all expectations involved are assumed to be finite (hence exist in the first place).]

6. **Linear approximation of one r.v. by another** (10 points)

Given two (jointly distributed) random variables $(X, Y)$, we want to approximate $X$ by a linear function of $Y$. That is we like to find constant coefficients $a, b \in \mathbb{R}$ such that the following mean squared error is minimized:

$$E[(X - (a \cdot Y + b))^2]. \quad (4)$$

Try to express $a, b$ in terms of expectations and (co)variances of $X$ and $Y$ and prove your result.

7. **Approximation of one r.v. by another** (10 points)

Given two random variables $(X, Y)$, we want to approximate $X$ by a function (not necessarily linear this time) of $Y$ such that the following mean squared error is minimized:

$$E[(X - g(Y))^2]. \quad (5)$$

Use the definition of conditional expectation to show that $g(Y)$ is the conditional expectation $E[X|Y]$. **Hint:** Try to use the definition to show the identity:

$$E[(X - g(Y))^2] = E[(X - E[X|Y])^2] + E[(g(Y) - E[X|Y])^2].$$

Therefore, among all functions of $Y$, the conditional expectation $E[X|Y]$ “approximates” $X$ the best.

8. **Identities associated to conditional expectation** (10 points)

Verify, using joint density functions, the following important identities:

(a) $E[g(Y) h(X)|Y] = g(Y) E[h(X)]|Y]$. (Y acts as constant if it is given in the condition).

(b) $E[E[X|Y]] = E[X]$. (This is called *total probability formula*).

(c) $E[E[X|Y, Z]|Y] = E[X|Y]$. (This is called *iterated averaging*).

(d) $E[X|Y] = E[X]$ if $X, Y$ are independent. (The opposite is not true - can you try to think of an example?).

However, for (a) and (b), you should try yourself to prove using the definition of conditional expectation.