Primary Reading: Chapter 1 of Stark and Woods. Books on graduate level probability theory and measure theory may give a more complete and rigorous introduction to probability space.

Homework is due at the beginning of class on the due day!

Problems:

1. Probability space (10 points)
   A man has two coins in his pocket, one is a fair coin with a head and a tail; another is a special coin with two heads. The man picks out a coin randomly, tosses it and observes if he gets a head or a tail. After that, he puts the coin back to the pocket and repeats the procedure for one more time. Questions: 1. What are all possible outcomes of this experiment, i.e., the sample space $\Omega$? 2. What is the event space, i.e., the $\sigma$-field $\mathcal{F}$? 3. What is the probability measure $P$ on each event? 4. What is the probability that he will get a tail for the second toss if he gets a head for the first time?

2. Inferior and superior limits (10 points)
   Given a sequence of sets $A_i$, $i = 1, 2, \ldots, \infty$ in a $\sigma$-field $\mathcal{F}$, prove that
   \[ A_* = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j \subseteq \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j = A^*. \]
   Therefore, $A_*$ and $A^*$ are also elements in $\mathcal{F}$. If $A_* = A^*$, we usually denote them as $\lim_{i \to \infty} A_i = A_* = A^*$, called the limit of the sequence $A_i$.

3. Probability of limit set (10 points)
   Consider a probability space $(\Omega, \mathcal{F}, P)$. Show that, for every increasing sequence of sets $A_i \in \mathcal{F}$, $i = 1, 2, \ldots, \infty$, i.e., $A_i \subseteq A_{i+1}, \forall i$, we have
   \[ P(\lim_{i \to \infty} A_i) = \lim_{i \to \infty} P(A_i). \]
   (Hint: the key is to construct the limit set $\lim_{i \to \infty} A_i$ as a sequence of disjoint sets.)

4. Conditional probability (10 points)
   For a probability space $(\Omega, \mathcal{F}, P)$, prove that for all $B \in \mathcal{F}$, if $P(B) > 0$, then the conditional probability $P(\cdot \mid B)$ is also a probability measure defined on $\mathcal{F}$. We usually call the induced probability space $(\Omega, \mathcal{F}, P(\cdot \mid B))$ as the conditional probability space associated to $B$.

5. Independence and conditional independence with complement set (10 points)
   Consider a probability space $(\Omega, \mathcal{F}, P)$.
   (a) Under what condition, are $A, A^c \in \mathcal{F}$ independent events?
   (b) If $C \in \mathcal{F}$ and $P(C) > 0$, under what condition, are $A, A^c \in \mathcal{F}$ conditionally independent given $C$?

6. Conditional independence (20 points)
   Given a probability space $(\Omega, \mathcal{F}, P)$ and $A, B, C \in \mathcal{F}$, prove the following:
(a) If \( A, C \) are independent, prove that \( A^c \) and \( C \) are also independent.

(b) If \( A, B, C \) are (jointly) independent and \( P(C) > 0 \), prove that \( A, B \) are conditionally independent given \( C \).

(c) Suppose \( A, B \) are independent and \( C = (A - B) \cup (B - A) \). Prove that in general \( A, B \) are however not conditionally independent given \( C \).

(d) Suppose \( A, C \) are independent and \( B = A^c \). Hence \( B, C \) are independent too (according to (a)). Show that, however, in general \( A, B \) are not conditionally independent given \( C \).

(Notes: be cautious about the words “in general” in the above questions.)

7. **Sets that are not measurable** (20 points)

Consider the closed interval \([0, 1]\) and its Borel measure (on Borel sets) introduced in class. Follow the steps below to prove constructed sets are not measurable (hence are not Borel sets):

(a) Introduce an equivalence relation to numbers in \([0, 1]\): We say that \( x \) and \( y \) are equivalent if and only if \( x - y \) is a rational number. Using this relation, we may partition the set \([0, 1]\) into many equivalence classes: Each class is a subset of \([0, 1]\) which contains all and only the numbers that are equivalent. Prove that every two equivalence classes are disjoint. [Notes: Based on the fact that the union of a countable number of countable sets is also countable, one can further show that there are uncountably many such equivalence classes. Here you may need to know that the set of rational numbers is countable but that of real numbers is not countable.]

(b) Construct a set \( S \) by choosing one (and only one) element from each of the equivalence classes. According to (a), elements in \( S \) have the properties that \( \forall x, y \in S, x - y \) must be irrational, and there are uncountably many elements in \( S \). For every rational number \( q \in [0, 1] \), we may construct a new set \( S_q \) of \( \{ x + q(\text{mod} \ 1) \mid x \in S \} \) as a subset of \([0, 1]\). Prove that, for two rational numbers \( p, q \) in \([0, 1]\), \( S_p \) and \( S_q \) are disjoint if \( p \neq q \).

(c) Prove that every number \( x \in [0, 1] \) must be in one of the \( S_q \) for some rational number \( q \in [0, 1] \). Let \( I = Q \cap [0, 1] \). Combining (b) and (c), we have: \( \{ S_i \}_{i \in I} \) is a countable sequence of disjoint sets and \( \bigcup_{i \in I} S_i = [0, 1] \).

(d) Use the fact that the Borel measure is translation-invariant to prove that every so defined \( S_i \) is not measurable. [Hint: Proof by contradiction.]

This problem serves as an example to many things that we have mentioned in class: 1. Not every subset of \([0, 1]\) is necessarily in the Borel \( \sigma \)-field; 2. Why in general we do not allow a \( \sigma \)-field to be closed under the union (hence intersection) of uncountable number of sets - the set \( S \) is the union of uncountable points, each of which is a valid Borel set.